Part A: MULTIPLE STATE MODELS

A1. MARKOV CHAIN PROBABILITIES

Transition probability:

${}_tp_x^{ij}$ is the probability that a life aged x who is currently in state i will be in state j at time t.

${}_tp_x^{ii}$ is the probability that a life aged x who is currently in state i will be in state i at time t.

${}_tp_x^{\overline{ii}}$ is the probability that a life aged x who is currently in state i will be in state i until time t.

$${}_tp_x^{\overline{ii}} \le {}_tp_x^{ii}$$

$${}_0p_x^{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Chapman-Kolmogorov equations: $${}_kp_x^{ij} = \sum_s {}_rp_x^{is}\ {}_{k-r}p_{x+r}^{sj}$$

Forces of transition: $$\mu_{x+t}^{ij} \quad \rightarrow \quad {}_tp_x^{\overline{ii}} = e^{-\int_0^t \sum_{j\neq i} \mu_{x+s}^{ij} ds}$$

Kolmogorov's forward equations: $$\frac{d}{dt}{}_tp_x^{ij} = \frac{{}_{t+h}p_x^{ij} - {}_tp_x^{ij}}{h} = \sum_{k\neq j}\left({}_tp_x^{ik}\ \mu_{x+t}^{kj} - {}_tp_x^{ij}\ \mu_{x+t}^{jk}\right)$$

A2. ESTIMATION OF TRANSITION INTENSITIES

Define the following: : $T^{(i)}$ is the total waiting time in State i between ages x and $x+1$

D^{ij} is the number of direct transitions from State i to State j between ages x and $x+1$

MLE of μ_x^{ij}: $$\hat{\mu}_x^{ij} = \frac{D^{ij}}{T^{(i)}}$$

Variance: $$\mathrm{Var}\left(\hat{\mu}_x^{ij}\right) \approx \frac{D^{ij}}{\left(T^{(i)}\right)^2}$$

A3. INSURANCE & ANNUITIES

Insurance functions:

$$A_x^{ij} = \sum_{k=0}^{\infty} v^{k+1}\left({}_{k+1}p_x^{ij} - {}_kp_x^{ij}\right) \qquad \bar{A}_x^{ij} = \int_0^\infty v^t \sum_{k\neq j} {}_tp_x^{ik}\ \mu_{x+t}^{kj} dt$$

Annuity functions:

$$\ddot{a}_x^{ij} = \sum_{k=0}^{\infty} v^k\,{}_kp_x^{ij} \qquad \bar{a}_x^{ij} = \int_0^\infty v^t\,{}_tp_x^{ij} dt$$

Relations:

$$A_x^{ij} = A_{x:\overline{n}|}^{ij} + \sum_k v^n\,{}_np_x^{ik}\ A_{x+n}^{kj} \qquad \bar{A}_x^{ij} = \bar{A}_{x:\overline{n}|}^{ij} + \sum_k v^n\,{}_np_x^{ik}\ \bar{A}_{x+n}^{kj}$$

$$\ddot{a}_x^{ij} = \ddot{a}_{x:\overline{n}|}^{ij} + \sum_k v^n\,{}_np_x^{ik}\ \ddot{a}_{x+n}^{kj} \qquad \bar{a}_x^{ij} = \bar{a}_{x:\overline{n}|}^{ij} + \sum_k v^n\,{}_np_x^{ik}\ \bar{a}_{x+n}^{kj}$$

Continuous sojourn annuity: $\bar{a}^{\overline{ii}}_{x:\overline{n}|} = \int_0^n v^t \, {}_tp^{\overline{ii}}_x dt$

Woolhouse formula, 2 terms: $\ddot{a}^{(m)ij}_x \approx \bar{a}^{ij}_x$ $\quad$ $\ddot{a}^{(m)ii}_x \approx \bar{a}^{ii}_x + \frac{1}{2m}$

Woolhouse formula, 3 terms: $\ddot{a}^{(m)ij}_x \approx \bar{a}^{ij}_x - \frac{1}{12m^2}\mu^{ij}_x$ $\quad$ $\ddot{a}^{(m)ii}_x \approx \bar{a}^{ii}_x + \frac{1}{2m} + \frac{1}{12m^2}\left(\delta + \sum_{j\neq i}\mu^{ij}_x\right)$

$\ddot{a}^{(m)ij}_x \approx \ddot{a}^{ij}_x + \frac{m^2-1}{12m^2}\mu^{ij}_x$ $\quad$ $\ddot{a}^{(m)ii}_x = \ddot{a}^{ii}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\delta + \sum_{j\neq i}\mu^{ij}_x\right)$

A4. PREMIUMS & RESERVES

Let: ${}_tV^{(i)}$ be the reserve at time t if the life is in state i at that time.

$I^{(i)}$ be the premium (calculated using the equivalence principle) less benefit while the life is in state i.

$b^{(ij)}$ be the (usually one-time) benefit when the life transitions from state i to state j.

Recursive formula:

$$\left({}_kV^{(i)} + I^{(i)}\right)(1+i) = \sum_j \left(b^{(ij)} + {}_{k+1}V^{(j)}\right) p^{ij}_{x+k}$$

$$\left({}_kV^{(i)} + sI^{(i)}\right)(1+i)^s = \sum_j \left(b^{(ij)} + {}_{k+s}V^{(j)}\right) {}_sp^{ij}_{x+k} \qquad \text{Where } 0 < s < 1.$$

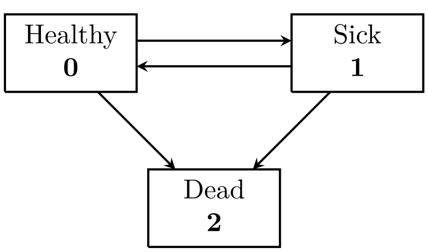
Thiele's differential equations:

$$\frac{d}{dt}{}_tV^{(i)} = \frac{{}_{t+h}V^{(i)} - {}_tV^{(i)}}{h} = {}_tV^{(i)}\left(\delta_t + \sum_{j\neq i}\mu^{ij}_{x+t}\right) + I^{(i)}_t - \sum_{j\neq i}\left(b^{(ij)}_t + {}_tV^{(j)}\right)\mu^{ij}_{x+t} \qquad \text{Euler's forward method}$$

$$\frac{d}{dt}{}_tV^{(i)} = \frac{{}_tV^{(i)} - {}_{t-h}V^{(i)}}{h} = {}_tV^{(i)}\left(\delta_t + \sum_{j\neq i}\mu^{ij}_{x+t}\right) + I^{(i)}_t - \sum_{j\neq i}\left(b^{(ij)}_t + {}_tV^{(j)}\right)\mu^{ij}_{x+t} \qquad \text{Euler's backward method}$$

A5. DISABILITY INCOME INSURANCE MODEL

Discrete DII model:



Transition matrix:

$$\mathbf{P}^{(t)} = \begin{pmatrix} p^{00}_{x+t} & p^{01}_{x+t} & p^{02}_{x+t} \\ p^{10}_{x+t} & p^{11}_{x+t} & p^{12}_{x+t} \\ p^{20}_{x+t} & p^{21}_{x+t} & p^{22}_{x+t} \end{pmatrix} = \begin{pmatrix} p^{00}_{x+t} & p^{01}_{x+t} & p^{02}_{x+t} \\ p^{10}_{x+t} & p^{11}_{x+t} & p^{12}_{x+t} \\ 0 & 0 & 1 \end{pmatrix}$$

Transition Probabilities:

$$ {}_2p_x^{00} = p_x^{00} p_{x+1}^{00} + p_x^{01} p_{x+1}^{10} + p_x^{02} p_{x+1}^{20} $$

$$ {}_2p_x^{\overline{00}} = p_x^{00} p_{x+1}^{00} $$

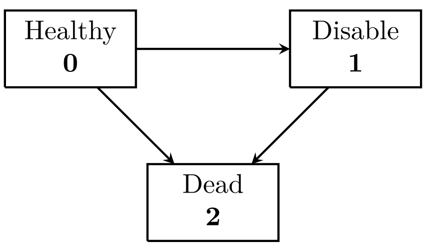
$$ {}_2p_x^{10} = \begin{pmatrix} p_x^{10} & p_x^{11} & p_x^{12} \end{pmatrix} \begin{pmatrix} p_{x+1}^{00} \\ p_{x+1}^{10} \\ 0 \end{pmatrix} $$

$$ {}_3p_x^{10} = \begin{pmatrix} p_x^{10} & p_x^{11} & p_x^{12} \end{pmatrix} \begin{pmatrix} p_{x+1}^{00} & p_{x+1}^{01} & p_{x+1}^{02} \\ p_{x+1}^{10} & p_{x+1}^{11} & p_{x+1}^{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{x+2}^{00} \\ p_{x+2}^{10} \\ 0 \end{pmatrix} $$

$$ {}_4p_x^{10} = \begin{pmatrix} p_x^{10} & p_x^{11} & p_x^{12} \end{pmatrix} \begin{pmatrix} p_{x+1}^{00} & p_{x+1}^{01} & p_{x+1}^{02} \\ p_{x+1}^{10} & p_{x+1}^{11} & p_{x+1}^{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{x+2}^{00} & p_{x+2}^{01} & p_{x+2}^{02} \\ p_{x+2}^{10} & p_{x+2}^{11} & p_{x+2}^{12} \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{x+3}^{00} \\ p_{x+3}^{10} \\ 0 \end{pmatrix} $$

A6. PERMANENT DISABILITY MODEL

Continuous PD model:



Transition Intensities: μ_{x+t}^{01} μ_{x+t}^{02} μ_{x+t}^{12}

Transition Probabilities:

$$ {}_tp_x^{00} = e^{-\int_0^t (\mu_{x+s}^{01} + \mu_{x+s}^{02}) ds} \qquad {}_tp_x^{01} = \int_0^t {}_sp_x^{00}\ \mu_{x+s}^{01}\ {}_{t-s}p_x^{11} ds \qquad {}_tp_x^{02} = 1 - {}_tp_x^{00} - {}_tp_x^{01} $$

$$ {}_tp_x^{10} = 0 \qquad {}_tp_x^{11} = e^{-\int_0^t \mu_{x+s}^{12} ds} \qquad {}_tp_x^{12} = 1 - {}_tp_x^{11} $$

$$ {}_tp_x^{20} = 0 \qquad {}_tp_x{}^{21} = 0 \qquad {}_tp_x{}^{22} = 1 $$

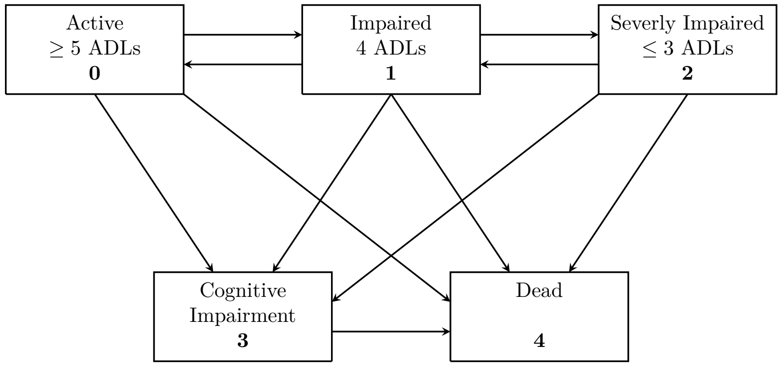
EPV of a benefit of 1 paid continuously while sick to a life currently aged x and healthy, with a waiting period of w years, and a policy term of $n > w$ years:

$$ \bar{a}_{x:\overline{n}|}^{01} = \int_0^n v^t\, {}_tp_x^{00}\ \mu_{x+t}^{01}\ \bar{a}_{x+t:\overline{n-t}|}^{\overline{11}} dt \qquad \text{If } w = 0. $$

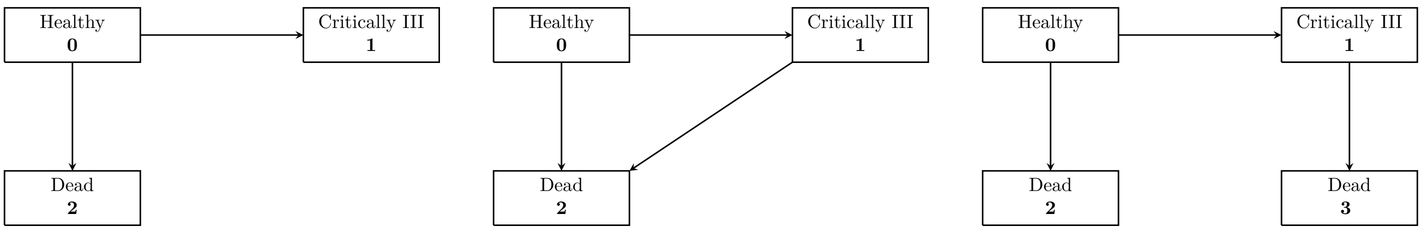
$$ EPV = \int_0^{n-w} v^t\, {}_tp_x^{00}\ \mu_{x+t}^{01} \left(\bar{a}_{x+t:\overline{n-t}|}^{\overline{11}} - \bar{a}_{x+t:\overline{w}|}^{\overline{11}} \right) dt \qquad \text{If } w > 0. $$

A7. OTHER MARKOV CHAIN MODELS

Long Term Care Insurance:

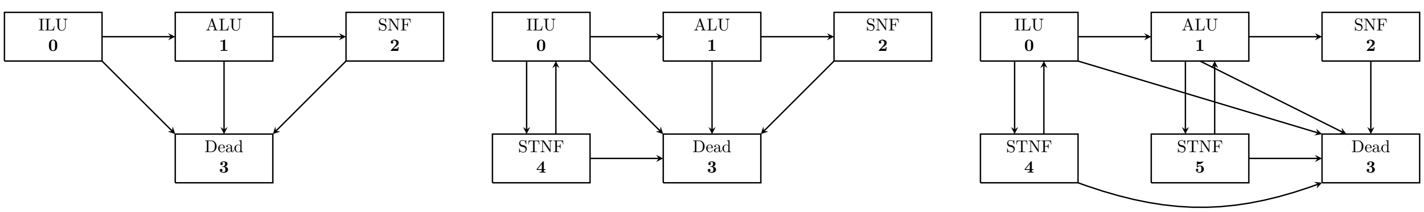


Critical Illness and Chronic Illness Insurance:



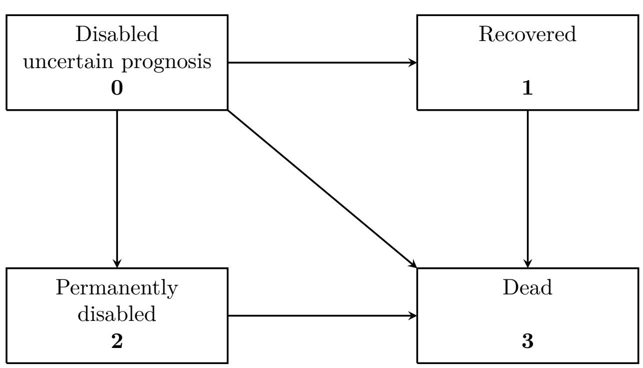
(a) Model 1 (B) Model 2 (C) Model 3

Continuing Care Retirement Communities:



CCRC with no reentry CCRC with temporary SNF CCRC with temporary SNF from ILU and ALU

Structured Settlements:



Reviewable annuity model

Part B: MULTIPLE DECREMENT MODELS

B1. MULTIPLE DECREMENT PROBABILITIES

Forces of decrement: $\mu_{x+s}^{(\tau)} = \sum_j \mu_{x+s}^{(j)} \quad \rightarrow \quad {}_tp_x^{(\tau)} = e^{-\int_0^t \mu_{x+s}^{(\tau)} ds}$

$\rightarrow \quad {}_tq_x^{(\tau)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(\tau)} ds \quad \rightarrow \quad \mu_{x+t}^{(\tau)} = \frac{\frac{d}{dt}\,{}_tq_x^{(\tau)}}{{}_tp_x^{(\tau)}}$

Multiple decrement probabilities: ${}_tp_x^{(j)}$ does not exist.

${}_tq_x^{(j)} = \int_0^t {}_sp_x^{(\tau)} \mu_{x+s}^{(j)} ds \quad \rightarrow \quad {}_tq_x^{(\tau)} = \sum_j {}_tq_x^{(j)} \quad \rightarrow \quad \mu_{x+t}^{(j)} = \frac{\frac{d}{dt}\,{}_tq_x^{(j)}}{{}_tp_x^{(\tau)}}$

Associated single decrement probabilities:

${}_tp_x^{\prime(j)} = e^{-\int_0^t \mu_{x+s}^{(j)} ds} \quad \rightarrow \quad {}_tp_x^{(\tau)} = \prod_j {}_tp_x^{(j)}$

${}_tq_x^{\prime(j)} = \int_0^t {}_sp_x^{\prime(j)} \mu_{x+s}^{(j)} ds \quad \rightarrow \quad \mu_{x+t}^{(j)} = \frac{\frac{d}{dt}\,{}_tq_x^{\prime(j)}}{{}_tp_x^{\prime(j)}}$

Formulas: ${}_{t+u}p_x^{(\tau)} = {}_tp_x^{(\tau)}\,{}_up_{x+t}^{(\tau)}$

${}_{t+u}q_x^{(\tau)} = {}_tq_x^{(\tau)} + {}_tp_x^{(\tau)}\,{}_uq_{x+t}^{(\tau)}$

${}_{t+u}q_x^{(j)} = {}_tq_x^{(j)} + {}_tp_x^{(\tau)}\,{}_uq_{x+t}^{(j)}$

${}_{t|u}q_x^{(j)} = {}_tp_x^{(\tau)}\,{}_uq_{x+t}^{(j)}$

Multiple decrement table: $l_{x+t}^{(\tau)} = l_x^{(\tau)}\ {}_tp_x^{(\tau)} \quad \rightarrow \quad d_{x+t}^{(\tau)} = l_x^{(\tau)}\,{}_tp_x^{(\tau)} q_{x+t}^{(\tau)}$

$\rightarrow \quad d_{x+t}^{(j)} = l_x^{(\tau)}\,{}_tp_x^{(\tau)} q_{x+t}^{(j)}$

B2. FRACTIONAL AGE ASSUMPTIONS

CFD between integral ages, for $0 < s+t \leq 1$:

$${}_sq_x^{(j)} = \frac{q_x^{(j)}}{q_x^{(\tau)}}\left(1 - \left(p_x^{(\tau)}\right)^s\right)$$

$${}_sp_{x+t}^{\prime(j)} = \left({}_sp_{x+t}^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}} \quad \rightarrow \quad q_x^{(j)} = q_x^{(\tau)}\left(\frac{\log p_x^{\prime(j)}}{\log p_x^{(\tau)}}\right)$$

UDD between integral ages in multiple decrement table, for $0 < s+t \leq 1$:

$${}_sq_x^{(j)} = s q_x^{(j)}$$

$${}_sp_{x+t}^{\prime(j)} = \left({}_sp_{x+t}^{(\tau)}\right)^{q_x^{(j)}/q_x^{(\tau)}} \quad \rightarrow \quad q_x^{(j)} = q_x^{(\tau)}\left(\frac{\log p_x^{\prime(j)}}{\log p_x^{(\tau)}}\right)$$

UDD between integral ages in associated single decrement table, for $0 < s+t \leq 1$:

$${}_sq_x{}^{(1)} = s q_x^{\prime(1)} - \frac{s^2}{2} q_x^{\prime(1)} q_x^{\prime(2)}$$ Two decrements

$${}_sq_x{}^{(1)} = s q_x^{\prime(1)} - \frac{s^2}{2} q_x^{\prime(1)}\left(q_x^{\prime(2)} + q_x^{\prime(3)}\right) + \frac{s^3}{3} q_x^{\prime(1)} q_x^{\prime(2)} q_x^{\prime(3)}$$ Three decrements

B3. ESTIMATION OF DECREMENT PROBABILITIES

Define the following: $T^{(0)}$ is the total waiting time in State 0 between ages x and $x+1$

D^{0j} is the number of direct transitions from State 0 to State j between ages x and $x+1$

MLE of $\mu_x^{(j)}$: $\hat{\mu}_x^{(j)} = \frac{D^{0j}}{T^{(0)}}$

Variance: $\text{Var}\left(\hat{\mu}_x^{0j}\right) \approx \frac{D^{0j}}{\left(T^{(0)}\right)^2}$

Assumption: $\mu_x^{(j)}$ is constant between ages x and $x+1$

MLE of $q_x'^{(j)}$: $\hat{q}_x'^{(j)} = 1 - e^{-\hat{\mu}_x^{(j)}}$

Variance: $\text{Var}\left(\hat{q}_x'^{(j)}\right) \approx \left(1 - \hat{q}_x^{(j)}\right)^2 \frac{D^{0j}}{\left(T^{(0)}\right)^2}$

B4. TRANSITIONS AT EXACT AGES

For a double decrement model:

Decrement 1 occurs uniformly throughout the year	Decrement 2 occurs only at the **beginning of the year**	$q_x^{(1)} = p_x'^{(2)} q_x'^{(1)}$ $q_x^{(2)} = q_x'^{(2)}$
	Decrement 2 occurs only at the **end of the year**	$q_x^{(1)} = q_x'^{(1)}$ $q_x^{(2)} = p_x'^{(1)} q_x'^{(2)}$
	Decrement 2 occurs only at the **middle of the year**	$q_x^{(1)} = {}_{0.5}q_x^{(1)} + {}_{0.5}p_x^{(\tau)}\,{}_{0.5}q_{x+0.5}^{(1)} = {}_{0.5}q_x'^{(1)} + {}_{0.5}p_x'^{(1)}\,{}_{0.5}p_x'^{(2)}\,{}_{0.5}q_{x+0.5}'^{(1)}$ $q_x^{(2)} = {}_{0.5}q_x^{(2)} = {}_{0.5}p_x'^{(1)}\,{}_{0.5}q_x'^{(2)}$ Note that ${}_{0.5}q_{x+0.5}^{(2)} = 0$.

B5. INSURANCE AND ANNUITIES, PREMIUMS & RESERVES

Insurance functions: $A_x^{(j)} = \sum_{k=0}^{\infty} v^{k+1}\,{}_kp_x^{(\tau)} q_{x+k}^{(j)}$ $\quad \bar{A}_x^{(j)} = \int_0^{\infty} v^t\,{}_tp_x^{(\tau)} \mu_{x+t}^{(j)} dt$

Annuity functions: $\ddot{a}_x^{(\tau)} = \sum_{k=0}^{\infty} v^k\,{}_kp_x^{(\tau)}$ $\quad \bar{a}_x^{(\tau)} = \int_0^{\infty} v^t\,{}_tp_x^{(\tau)} dt$

Recursive formula: $\left({}_kV^g + G - e\right)(1+i) = \sum_j \left(b^{(j)} + E^{(j)}\right) q_{x+k}^{(j)} + {}_{k+1}V^g\ p_{x+k}^{(\tau)}$

$\left({}_kV^g + G - e\right)(1+i)^s = \sum_j \left(b^{(j)} + E^{(j)}\right) {}_sq_{x+k}^{(j)} + {}_{k+s}V^g\ {}_sp_{x+k}^{(\tau)}$ where $0 < s < 1$.

B6. NOTATION

Multiple decrement model	Multiple state model	Statistics
${}_tp_x^{(\tau)}$	${}_tp_x^{00}$	$\Pr\left(T_x > t\right)$

${}_tq_x^{(\tau)}$	$\sum_j {}_tp_x{}^{0j}$	$\Pr(T_x \le t)$
${}_tq_x^{(j)}$	${}_tp_x^{0j}$	$\Pr(T_x \le t, J = j)$
${}_tq_x'^{(j)}$	N/A	$\Pr\left(T_x^{(j)} \le t\right)$

Part C: MULTIPLE LIFE MODELS

C1. MULTIPLE LIFE PROBABILITIES

Joint life probabilities: ${}_tp_{xy} = \Pr(T_{xy} > t) = \Pr(T_x > t, T_y > t) = \Pr(\text{Both lives survive } t \text{ years}) = e^{-\int_0^t \mu_{x+s:y+s}\ dt}$

${}_tq_{xy} = \Pr(T_{xy} \le t) = \Pr(\text{At least one life cannot survive } t \text{ years})$

Last survivor probabilities: ${}_tq_{\overline{xy}} = \Pr(T_{\overline{xy}} \le t) = \Pr(T_x \le t)\Pr(T_y \le t) = \Pr(\text{Both lives die within } t \text{ years})$

${}_tp_{\overline{xy}} = \Pr(T_{\overline{xy}} > t) = \Pr(\text{At least one life survive } t \text{ years})$

Formulas:

${}_{t+u}p_{xy} = {}_tp_{xy}\ {}_up_{x+t:y+t}$

${}_{t|u}q_{xy} = {}_tp_{xy} - {}_{t+u}p_{xy} = {}_tp_{xy}\ {}_uq_{x+t:y+t}$

${}_{t+u}p_{\overline{xy}} \ne {}_tp_{\overline{xy}}\ {}_up_{\overline{x+t:y+t}}$

${}_{t|u}q_{\overline{xy}} = {}_tp_{\overline{xy}} - {}_{t+u}p_{\overline{xy}} \ne {}_tp_{\overline{xy}}\ {}_uq_{\overline{x+t:y+t}}$

Contingent probabilities: ${}_tq_{\overset{1}{x}y} = \Pr(T_x < T_y, T_x \le t) = \Pr((x) \text{ dies first and within } t \text{ years})$

${}_tq_{x\overset{1}{y}} = \Pr(T_y < T_x, T_y \le t) = \Pr((y) \text{ dies first and within } t \text{ years})$

${}_tq_{\overset{2}{x}y} = \Pr(T_y < T_x \le t) = \Pr((x) \text{ dies second and within } t \text{ years})$

${}_tq_{x\overset{2}{y}} = \Pr(T_x < T_y \le t) = \Pr((y) \text{ dies second and within } t \text{ years})$

If the future lifetimes of (x) and (y) are independent:

$\mu_{x+t:y+t} = \mu_{x+t} + \mu_{y+t}$

${}_tp_{xy} = {}_tp_x\ {}_tp_y$

${}_tq_{\overset{1}{x}y} = \int_0^t {}_sp_x\ {}_sp_y\ \mu_{x+t} ds$

${}_tq_{\overline{xy}} = {}_tq_x\ {}_tq_y$

${}_tq_{\overset{2}{x}y} = \int_0^t {}_sp_x\ {}_sq_y\ \mu_{x+t} ds$

Relations:

${}_tp_{xy} + {}_tq_{xy} = 1$

${}_tq_{xy} = {}_tq_{\overset{1}{x}y} + {}_tq_{x\overset{1}{y}}$

${}_tq_{\overline{xy}} + {}_tp_{\overline{xy}} = 1$

${}_tq_{\overline{xy}} = {}_tq_{\overset{2}{x}y} + {}_tq_{x\overset{2}{y}}$

${}_tp_x + {}_tp_y = {}_tp_{xy} + {}_tp_{\overline{xy}}$

${}_tq_x = {}_tq_{\overset{1}{x}y} + {}_tq_{\overset{2}{x}y}$

${}_tq_x + {}_tq_y = {}_tq_{xy} + {}_tq_{\overline{xy}}$

${}_tq_y = {}_tq_{x\overset{1}{y}} + {}_tq_{x\overset{2}{y}}$

C2. INSURANCE, ANNUITIES, PREMIUMS & RESERVES

Insurance functions:

$\bar{A}_{xy}$ or A_{xy} = EPV of 1 per year until the first death

$\bar{A}_{\overline{xy}}$ or $A_{\overline{xy}}$ = EPV of 1 per year until the second death

$\bar{A}_{\overset{1}{x}y}$ or $A_{\overset{1}{x}y}$ = EPV of 1 paid upon $(x)'s$ death if (x) dies first

$\bar{A}_{\overset{2}{x}y}$ or $A_{\overset{2}{x}y}$ = EPV of 1 paid upon $(x)'s$ death if (x) dies second

Relations:

$\bar{A}_x + \bar{A}_y = \bar{A}_{xy} + \bar{A}_{\overline{xy}}$ $\qquad$ $A_x + A_y = A_{xy} + A_{\overline{xy}}$

$\bar{A}_{xy} = \bar{A}_{\overset{1}{x}y} + \bar{A}_{x\overset{1}{y}}$ $\qquad$ $A_{xy} = A_{\overset{1}{x}y} + A_{x\overset{1}{y}}$

$\bar{A}_{\overline{xy}} = \bar{A}_{\overset{2}{x}y} + \bar{A}_{x\overset{2}{y}}$ $\qquad$ $A_{\overline{xy}} = A_{\overset{2}{x}y} + A_{x\overset{2}{y}}$

$\bar{A}_x = \bar{A}_{\overset{1}{x}y} + \bar{A}_{\overset{2}{x}y}$ $\qquad$ $A_x = A_{\overset{1}{x}y} + A_{\overset{2}{x}y}$

Annuity functions:

$\bar{a}_{xy}$ or $\ddot{a}_{xy}$ = EPV of 1 per year until the first death

$\bar{a}_{\overline{xy}}$ or $\ddot{a}_{\overline{xy}}$ = EPV of 1 per year until the second death

$\bar{a}_{x|y}$ or $\ddot{a}_{x|y}$ = EPV of 1 per year until (y) dies – payments begin only after (x) dies

Relations:

$\bar{a}_x + \bar{a}_y = \bar{a}_{xy} + \bar{a}_{\overline{xy}}$ $\qquad$ $\ddot{a}_x + \ddot{a}_y = \ddot{a}_{xy} + \ddot{a}_{\overline{xy}}$

$\bar{a}_{x|y} = \bar{a}_y - \bar{a}_{xy}$ $\qquad$ $\ddot{a}_{x|y} = \ddot{a}_y - \ddot{a}_{xy}$

Insurance to annuities:

$\bar{A}_{xy} = \dfrac{1 - \bar{a}_{xy}}{\delta}$ $\qquad$ $A_{xy} = \dfrac{1 - \ddot{a}_{xy}}{d}$

$\bar{A}_{\overline{xy}} = \dfrac{1 - \bar{a}_{\overline{xy}}}{\delta}$ $\qquad$ $A_{\overline{xy}} = \dfrac{1 - \ddot{a}_{\overline{xy}}}{d}$

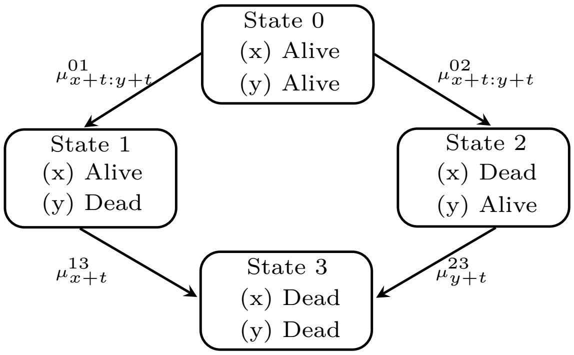
Covariance:

$Cov(v^{T_{xy}}, v^{T_{\overline{xy}}}) = (\bar{A}_x - \bar{A}_{xy})(\bar{A}_y - \bar{A}_{xy})$

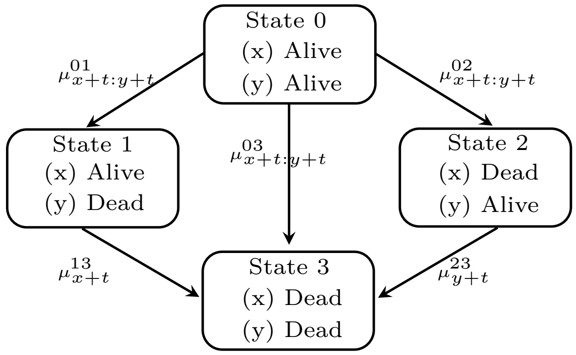
$Cov(\bar{a}_{\overline{T_{xy}}|}, \bar{a}_{\overline{T_{\overline{xy}}}|}) = (\bar{a}_x - \bar{a}_{xy})(\bar{a}_y - \bar{a}_{xy})$

C3. MULTIPLE LIFE MODELS

The "classical" multiple life model:



The "classical" common shock model:



Notation and formulas, assuming the **"classical" multiple life model**:

Multiple life model	Multiple state model	Statistics
${}_tp_{xy}$	${}_tp_x^{00}$	$\Pr(T_x > t, T_y > t)$
${}_tq_{xy}$	${}_tp_x^{01} + {}_tp_x^{02} + {}_tp_x^{03}$	$\Pr(T_x > t, T_y \le t) + \Pr(T_x \le t, T_y > t) + \Pr(T_x \le t, T_y \le t)$
${}_tq_{\overline{xy}}$	${}_tp_x^{03}$	$\Pr\left(T_x \le t, T_y \le t\right)$
${}_tp_{\overline{xy}}$	${}_tp_x^{00} + {}_tp_x^{01} + {}_tp_x^{02}$	$\Pr\left(T_x > t, T_y \le t\right) + \Pr\left(T_x \le t, T_y > t\right) + \Pr\left(T_x > t, T_y > t\right)$
${}_tp_x$	${}_tp_{xy}^{00} + {}_tp_{xy}^{01}$	$\Pr\left(T_x > t\right)$
${}_tq_x$	${}_tp_{xy}^{02} + {}_tp_{xy}^{03}$	$\Pr\left(T_x \le t\right)$
${}_tp_y$	${}_tp_{xy}^{00} + {}_tp_{xy}^{02}$	$\Pr\left(T_y > t\right)$
${}_tq_y$	${}_tp_{xy}^{01} + {}_tp_{xy}^{03}$	$\Pr\left(T_y \le t\right)$
${}_tq_{\overset{1}{x}y}$	$\int_0^t {}_sp_{xy}^{00}\ \mu_{x+s:y+s}^{02} ds \ne {}_tp_{xy}^{02}$	$\Pr\left(T_x \le t, T_x < T_y\right)$
${}_tq_{x\overset{1}{y}}$	$\int_0^t {}_sp_{xy}^{00}\ \mu_{x+s:y+s}^{01} ds \ne {}_tp_{xy}^{01}$	$\Pr\left(T_y \le t, T_y < T_x\right)$
${}_tq_{\overset{2}{x}y}$	$\int_0^t {}_sp_{xy}^{00}\ \mu_{x+s:y+s}^{01}\ {}_{t-s}p_{x+s}^{13} ds \ne {}_tp_{xy}^{03}$	$\Pr\left(T_x \le t, T_x > T_y\right)$
${}_tq_{x\overset{2}{y}}$	$\int_0^t {}_sp_{xy}^{00}\ \mu_{x+s:y+s}^{02}\ {}_{t-s}p_{x+s}^{23} ds \ne {}_tp_{xy}^{03}$	$\Pr\left(T_y \le t, T_y > T_x\right)$

Part D: PROFIT ANALYSIS

D1. PROFIT TESTING

Recall: **Net premiums** are always calculated using the **equivalence principle**.

Equivalence principle → Net premiums are expected to cover benefits.

But **gross premiums** may not be calculated using the equivalence principle.

Equivalence principle → Gross premiums are expected to cover benefits and expenses.

Otherwise → Gross premiums are expected to cover benefits, expenses and profits (or losses).

Reserve assumptions: Assumptions imposed by government to determine reserves.

Pricing assumptions: Assumptions the company uses to determine premiums.

Profit test assumptions: Assumptions the company uses to conduct **profit testing**.

For traditional policies, assumptions are mortality (or decrements/transitions), interest rate, expenses, and lapse.

Premiums and reserves are pre-determined or "fixed".

Define the following:

$_tV$ is the reserve at time t

G_t is the premium collected at time t

E_t is the annual expenses incurred at time t

i_t is the interest rate earned on the investments from time t to time $t+1$

DB_t is the death benefit payable at time t

E_t^{DB} is the expenses associated with death benefit incurred at time t

Expected profit: $\text{Pr}_{t+1} = ({}_tV + G_t - E_t)(1+i_t) - \left(DB_{t+t} + E_{t+1}^{DB}\right) q_{x+k} - {}_{t+1}V\, p_{x+t}$

This formula is for alive-dead model and can be modified for MS/MD/ML models.

D2. PROFIT MEASURES

Profit vector: Pr_k — Pr_0 usually contains pre-contract expenses only.

Profit signature: $\Pi_k = {}_{k-1}p_x\, \text{Pr}_k$ — Note that $\Pi_0 = \text{Pr}_0$.

Internal rate of return: IRR — Such that $\sum_{k=0}^{n} \frac{\Pi_k}{(1+IRR)^k} = 0$.

Net present value: $NPV = \sum_{k=0}^{n} \frac{\Pi_k}{(1+r)^k}$ — Where r is the hurdle rate.

Partial NPV: $NPV(t) = \sum_{k=0}^{t} \frac{\Pi_k}{(1+r)^k}$ — Where r is the hurdle rate.

Discounted payback period: Smallest m — Such that $NPV(m) = \sum_{k=0}^{m} \frac{\Pi_k}{(1+r)^k} \geq 0$.

Profit margin: $M = \frac{NPV}{EPV(\text{Premiums})}$ — Where $EPV\,(\text{Premiums})$ is calculated using r, the hurdle rate.

Premiums can be determined using a desired profit margin.

Reserves can be determined using a process called **zeroization**.

D3. ACTUAL PROFIT & GAIN BY SOURCE

Expected profit: Profit(Expected interest, Expected expense, Expected mortality)

Actual profit: Profit(Actual interest, Actual expense, Actual mortality)

Overall gain: Overall gain = Actual profit - Expected profit

The usual order of **gain by source**: Interest → Expense → Mortality

Gain from interest = Profit(A.interest, E.expense, E.mortality) − Profit(E.interest, E.expense, E.mortality)

Gain from expense = Profit(A.interest, A.expense, E.mortality) − Profit(A.interest, E.expense, E.mortality)

Gain from mortality = Profit(A.interest, A.expense, A.mortality) − Profit(A.interest, A.expense, E.mortality)

Overall gain = Gain due to interest + Gain due to expense + Gain due to mortality

Part E: UNIVERSAL LIFE INSURANCE

E1. UL ACCOUNT VALUES

Define the following:

AV_t is the account value at time t

P_t is the premium collected at time t

EC_t is the expense charge deducted from the account value at time t

i_t^c is the credited interest rate earned on the investments time t to time $t+1$

CoI_t is the cost of insurance charge at time t

q_{x+t}^* is the CoI mortality rate

i^* is the CoI interest rate

FA is the face amount

SC_t is the surrender charge deducted from account value at time t

	Type A UL	**Type B UL**
Account Value	$(AV_t + P_t - EC_t)(1+i_t^c) = \frac{1+i_t^c}{1+i^*} q_{x+t}^* FA + \left(1 - \frac{1+i_t^c}{1+i^*} q_{x+t}^*\right) AV_{t+1}$	$(AV_t + P_t - EC_t)(1+i_t^c) = \frac{1+i_t^c}{1+i^*} q_{x+t}^* FA + AV_{t+1}$
Cost of Insurance	$CoI_{t+1} = \frac{1}{1+i^*} q_{x+t}^* (FA - AV_{t+1})$	$CoI_{t+1} = \frac{1}{1+i^*} q_{x+t}^* FA$
Additional Death Benefit	$ADB_{t+1} = FA - AV_{t+1}$	$ADB_{t+1} = FA$
Death Benefit	$DB_{t+1} = FA$	$DB_{t+1} = FA + AV_{t+1}$
Surrender Benefit	$SB_t = AV_{t+1} - SC_{t+1}$	$SB_t = AV_{t+1} - SC_{t+1}$

E2. UL CORRIDOR REQUIREMENT

Corridor factor: γ

Type A UL	Type B UL
1. Calculate AV_{t+1} using the above formula. 2. If $\gamma AV_{t+1} \leq FA$, good. 3. If $\gamma AV_{t+1} > FA$, replace FA with γAV_{t+1} in the formula and calculate AV_{t+1} again. The revised death benefit is γAV_{t+1}.	1. Calculate AV_{t+1} using the above formula. 2. If $\gamma AV_{t+1} \leq FA + AV_{t+1}$, good. 3. If $\gamma AV_{t+1} > FA + AV_{t+1}$, replace $FA + AV_{t+1}$ with γAV_{t+1} in the formula and calculate AV_{t+1} again. The revised death benefit is γAV_{t+1}.

E3. UL RESERVES

EPV of Type A UL death benefit: $(FA)A^{\;1}_{x+t:\overline{n-t}|}$

Type A UL no lapse guarantee reserve: ${}_tV^{NLG} = \max\left(0, (FA)A^{\;1}_{x+t:\overline{n-t}|} - AV_t\right)$

E4. UL PROFIT TESTING

Define the following:

E_t is the annual expenses incurred at time t

E_t^{DB} is the expenses associated with death benefit incurred at time t

E_t^{SB} is the expenses associated with surrender benefit incurred at time t

i_t is the interest rate earned on the investment from time t to time $t+1$

Expected profit: $\Pr_{t+1} = (AV_{t-1} + P_t - E_t)(1+i_t) - \left(DB_{t+1} + E_{t+1}^{DB}\right) q_{x+t}^{(d)} - \left(SB_{t+1} + E_{t+1}^{SB}\right) q_{x+t}^{(w)} - AV_{t+1} p_{x+t}^{(\tau)}$

Part F: EMBEDDED OPTIONS AND EQUITY-LINKED INSURANCE

F1. EMBEDDED OPTIONS

Four popular guarantees:

Guaranteed minimum death benefit (GMDB)

Guaranteed minimum accumulation/maturity benefit (GMAB/GMMB)

Guaranteed minimum withdrawal benefit (GMWB)

Guaranteed minimum income benefit (GMIB)

	Payoff and value of guarantee	Where
GMDB with a return of premium guarantee	Time-T payoff $= \max(0, K - S_T)$ Time-0 value $= \int_0^\infty P(K,t) f_T(t) dt$	T is the future lifetime of policyholder S_T is the AV at time T K is the amount invested $P(K,t)$ is the price of a K-strike put option expired at time t
Earnings-enhanced death benefit	Time-T payoff $= \alpha \max(0, S_T - K)$ Time-0 value $= \alpha \int_0^\infty C(K,t) f_T(t) dt$	T is the future lifetime of policyholder α is the guarantee percentage S_T is the AV at time T K is the amount invested $C(K,t)$ is the price of a K-strike call option expired at time t
GMAB/GMMB with a return of premium guarantee	Time-τ payoff $= \max(0, K - S_\tau)$ for $T^* > \tau$ Time-0 value $= P(K,\tau) \times \Pr(T^* > \tau)$	T^* is the future lifetime of policy τ is the guarantee period S_τ is the AV at time τ K is the amount invested $P(K,\tau)$ is the price of a K-strike put option expired at time τ

Define the following:

F_t is the policyholder's fund at time t

n is the maturity of the contract

P is the single premium

e is the expense charge

m is the management charge

S_t is the stock index, with $S_0 = 1$

$\xi = (1-e)(1-m)^{n-1}$

$h(t)$ is the option payoff at time t

$\nu(0,t) = E_0^Q[e^{-rt}h(t)]$

GMMB

Time 0 value:

$$\pi(0) = P\,{}_np_x\big(e^{-rn}\Phi(-d_2) - \xi\Phi(-d_1)\big)$$

$$d_1 = \frac{\ln\xi + (r + 0.5\sigma^2)n}{\sigma\sqrt{n}}$$

$$d_2 = d_1 - \sigma\sqrt{n}$$

Time t value (reserve) : $\pi(t) = P_{n-t}p_{x+t}\left(e^{-r(n-t)}\Phi(-d_2) - S_t\xi\Phi(-d_1)\right)$

$$d_1 = \frac{\ln S_t\xi + (r + 0.5\sigma^2)(n-t)}{\sigma\sqrt{n-t}}$$

$$d_2 = d_1 - \sigma\sqrt{n-t}$$

GMDB

Time 0 value (continuous): $\pi(0) = \int_0^n v(0,t)\,{}_tp_x\,\mu_{x+t}\,\mathrm{d}t$

Time 0 value (monthly): $\pi(0) = \sum_{t=0}^{12n-1} v(0,t/12)\,{}_{\frac{t-1}{12}}p_x\,{}_{\frac{1}{12}}q_{x+\frac{t-1}{12}}$

F2. HEDGING AND REBALANCING

GMMB Hedge Portfolio

Time t value $\pi(t) = {}_{n-t}p_{x+t}E_0^Q[e^{-r(n-t)}h(n)] = {}_{n-t}p_{x+t}\nu(t,n)$

Stock part ${}_{n-t}p_{x+t}\left(\frac{d}{dS_t}\nu(t,n)\right)S_t$

Zero-coupon bond part $\pi(t) - {}_{n-t}p_{x+t}\left(\frac{d}{dS_t}\nu(t,n)\right)S_t$

GMDB Hedge Portfolio

Time t value $\pi(t) = \int_0^{n-t}\nu(t,w+t)_wp_{x+t}\mu_{x+t+w}\mathrm{d}w$

Stock part $\int_0^{n-t}S_t\left(\frac{d}{dS_t}\nu(t,w+t)\right){}_wp_{x+t}\mu_{x+t+w}\mathrm{d}w$

Zero-coupon bond part $\int_0^{n-t}\left(\nu(t,w+t) - S_t\left(\frac{d}{dS_t}\nu(t,w+t)\right)\right){}_wp_{x+t}\mu_{x+t+w}\mathrm{d}w$

F3. PROFIT TESTING

Columns of profit test if company purchases options:

+	Initial expense charge in the first period, if any
+	Management charge
−	Expenses
−	Risk premium
+	Release of beginning reserve, if any
+	Interest, if any
−	Reserve per survivor at end of period, if any

Columns of profit test if company hedges internally:

+	Initial expense charge in the first period, if any
+	Management charge
−	Expenses
+	Release of beginning reserve, if any
+	Interest, if any
−	GMxB cost
−	Hedging/rehedging cost
−	Reserve per survivor at end of period, if any

F4. EQUITY-LINKED INSURANCE

Define the following:

F_t is the policyholder's fund at time t

P_t is the premium paid at time t

AP_t is allocated premium - the part of P_t that is invested in the fund

i_t^f is the interest rate earned on the fund from time t to time $t+1$

MC_t is the management charge deducted from the fund at time t

General Formula:

$$F_{t+1} = (F_t + AP_t)\left(1 + i_t^f\right) - MC_{t+1}$$

Define the following:

${}_tV$ is the reserve at time t

$UAP_t = P_t - AP_t$ is the unallocated premium

E_t is the expenses incurred at time t

i_t^a is the interest rate earned on the insurer's assets from time t to time $t+1$

DB_t is the death benefit payable at time t

CV_t is the cash value at time t

Expected profit:

$$\Pr_{t+1} = ({}_tV + UAP_t - E_t)(1 + i_t^a) + MC_{t+1} - (DB_{t+1} - F_{t+1})\, q_{x+t}^{(d)} - (CV_{t+1} - F_{t+1})\, q_{x+t}^{(w)} - {}_{t+1}V\, p_{x+t}^{00}$$

Part G: PENSIONS

G1. SERVICE TABLE

ALTAM service table: Death d_x

Disability i_x

Withdrawal w_x

Retirement r_x — The service table is a multiple decrement table.

G2. SALARY SCALE & REPLACEMENT RATIO

Salary rate: $\bar{S}_x$ — Salary rate at exact age x.

Salary scale: $s_x = \int_0^1 \bar{s}_{x+t}dt$ — Salary rate during the year $[x, x+1)$.

Approximation: $\bar{S}_x \approx S_{x-0.5}$

Salary: $S_y = S_x \frac{s_y}{s_x}$ — Salary received during the year $[y, y+1)$.

Replacement ratio: $R = \dfrac{\text{Pension income in the year after retirement}}{\text{Salary in the year before retirement}}$

Define the following: xe is the age the member enters the plan

xr is the age the member retires

$n_{xr} = xr - xe$ is the total number of years of service

$TPE_{xr} = S_{xe} + S_{xe+1} + \cdots + S_{xr-1}$ is the **total pensionable earnings**.

Final average salary: $S^F_{xr} = \dfrac{S_{xr-k} + \cdots + S_{xr-2} + S_{xr-1}}{k}$ — The average of salaries during the final k years.

Career average salary: $S^C_{xr} = \dfrac{TPE_{xr}}{n_{xr}}$ — The average of salaries during the whole career.

G3. DEFINED BENEFIT PLANS

Accrual rate of DB plan: α

DB plan annual benefit: $B = n_{xr}\alpha S^F_{xr}$ — Final average salary

$B = n_{xr}\alpha S^C_{xr} = \alpha\, TPE_{xr}$ — Career average salary

Sponsor of DB plan: If $B = n_{xr}\alpha S^F_{xr}$ is payable per year right after retirement and until the member dies, then the sponsor has to purchase an annuity worth $B\ddot{a}_{xr}$ for the member.
Early retirement benefits may be subject to a reduction factor RF_{xr}.

APV of DB plan: $APV = \sum_{xr} n_{xr} \times \alpha \times S_{xr}^F \times (1-RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Final average salary

$APV = \sum_{xr} \alpha \times TPE_{xr} \times (1-RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Career average salary

AL of DB plan, PUC method: $AL = \sum_{xr} n_x \times \alpha \times S_{xr}^F \times (1-RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Final average salary

$AL = \sum_{xr} n_x \times \alpha \times \frac{TPE_{xr}}{n_{xr}} \times (1-RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Career average salary

AL of DB plan, TUC method: $AL = \sum_{xr} n_x \times \alpha \times S_x^F \times (1-RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Final average salary

$AL = \sum_{xr} \alpha \times TPE_x \times (1-RF_{xr}) \times \ddot{a}_{xr} \times v^{xr-x} \times \frac{r_{xr}}{l_x}$ Career average salary

Normal contribution: $C = v\, p_x^{00}\ {}_1V - {}_0V +$ EPV(Benefits for mid-year exits)

No exit benefits, PUC method: $C = {}_0V\left(\frac{1}{n}\right)$ No exit benefits, and ${}_tV$ calculated using PUC method.

No exit benefits, TUC method: $C = {}_0V\left(\frac{S_{x+1}}{S_x}\frac{n+1}{n} - 1\right)$ No exit benefits, and ${}_tV$ calculated using TUC method.

G4. RETIREE HEALTHCARE PLANS

Premium for healthcare plan: $B(x,t)$ Premium paid at time t for a person aged x

Benefit premium annuity: $\ddot{a}_B(x,t) = \sum_{k=0}^{\infty} \frac{B(x+k,\ t+k)}{B(x,t)} v^k\, {}_kp_x$ This is an increasing annuity-due.

Simplification: $\ddot{a}_B(x,t) = \sum_{k=0}^{\infty} c^k(1+j)^k\ v^k\, {}_kp_x$ Where: $c = \frac{B(x+1,t)}{B(x,t)}$

$\ddot{a}_B(x,t) = \ddot{a}_{x@i^*}$ $1+j = \frac{B(x,\ t+1)}{B(x,t)}$

$1+i^* = \frac{1+i}{c(1+j)}$

EPV of retiree healthcare plan: $EPV = B(x,t)\ddot{a}_B(x,t)$

For someone retires at age x and time t.

AVTHB: $AVTHB = \sum_{t=0}^{65-x} B(x+t,t)\ddot{a}_B(x+t,t) \times v^t \times \frac{r_{x+t}}{l_x}$

$AVTHB = B(x,0)\sum_{t=0}^{65-x} \ddot{a}_{x+t@i^*} \times v^{*t} \times \frac{r_{x+t}}{l_x}$

Assume retirement must occur no later than age 65.

APBO: ${}_0V = \sum_{t=0}^{65-x} B(x+t,t)\ddot{a}_B(x+t,t) \times v^t \times \frac{r_{x+t}}{l_x} \times \frac{x-x_0}{x+t-x_0}$

Where x_0 is the age the employer begins working.

Normal cost: $C = vp_x\ {}_1V - {}_0V +$ EPV(Benefits for mid-year exits)

No exit benefits. $C = {}_0V(\frac{1}{n})$

Where n is the number of years of service