UTILITY THEORY

In economics, '**utility**' is the satisfaction that an individual obtains from a particular course of action. The Expected Utility Theorem states that a function, $U(w)$, can be constructed as representing an investor's utility of wealth, w, at some future date. Decisions are made in a manner to maximize the expected value of utility given the investor's particular beliefs about the probability of different outcomes.

We assume that people prefer more to less known as principle of non-satiation expressed as $U'(w) > 0$.

A **risk-averse** investor values an incremental increase in wealth less highly than an incremental decrease and will reject a fair gamble. The utility function condition is: $U''(w) < 0$

A **risk-seeking** investor values an incremental increase in wealth more highly than an incremental decrease and will seek a fair gamble. The utility function condition is: $U''(w) > 0$

For a risk-averse individual, the certainty equivalent is higher than the actual likelihood of the outcome, i.e. the individual would need to receive odds higher than expected to accept this gamble.

If the absolute value of the certainty equivalent decreases with increasing wealth, the investor is said to exhibit declining **absolute risk aversion**. If the absolute value of the certainty equivalent increases, the investor exhibits increasing absolute risk aversion. If the absolute value of the certainty equivalent decreases (increases) as a proportion of total wealth as wealth increases, the investor is said to exhibit a declining (increasing) relative risk aversion.

The example will explain the application of this.
Click here to see tab *Utility Theory* in the shared Excel file online.

RISK MEASURES

Commonly used measures of investment risk are:

- Variance of return
- Downside **semi-variance** of return
- **Shortfall probabilities**: A shortfall probability measures the probability of returns falling below a certain level
- **Value at Risk (VaR)**: VaR represents the maximum potential loss on a portfolio over a given future time period with a given degree of confidence, where the latter is normally expressed as $1-p$. So, for example, a 99% one-day VaR is the maximum loss on a portfolio over a one-day period with 99% confidence, i.e., there is a 1% probability of a greater loss.

We will show how to calculate them with the help of examples.
Click here to see tab *Risk Measures (1)* in the shared Excel file online.
Click here to see tab *Risk Measures (2)* in the shared Excel file online.

PORTFOLIO THEORY

Mean-variance portfolio theory, sometimes called modern portfolio theory (MPT), specifies a method for an investor to construct a portfolio that gives the maximum return for a specified risk, or the minimum risk for a specified return. This is the basis for portfolio theory.

In the example, we will see how to calculate the return on a portfolio and the variance given the correlation matrix between various assets.
Click here to see tab *Asset Returns* in the shared Excel file online.

CAPITAL ASSET PRICING MODEL

Portfolio theory can be applied by a single investor given their own estimates of security returns, variances and covariances. The **capital asset pricing model (CAPM)** introduces additional assumptions regarding the market and the behavior of other investors to allow the construction of an equilibrium model of prices in the whole market.

All rational investors will hold a combination of the risk-free asset and the portfolio of risky assets at the point where the straight line through the risk-free return (on the E-axis) touches the original efficient frontier. Because this is the portfolio held in different quantities by all investors, it must consist of all risky assets in proportion to their market capitalization. It is commonly called the '**market portfolio**'. The proportion of a particular investor's portfolio consisting of the market portfolio will be determined by their risk-return preference.

CAPM can be used to estimate the expected return on a financial security given its exposure to the various risk factors modeled. This return can then be used to discount projected future cash flows and so price the security and determine if it appears to be under-valued or over-valued.

CAPM equation is $E_i = r + \beta_i * (E_M - r)$, where

- E_i is the expected return on security i,
- r is the return on the risk-free asset,
- E_M is the expected return on the market portfolio,
- β_i is the beta factor of security i defined as Covariance $(R_i, R_M)/\text{Variance}_M$ and measures the systematic risk or the non-diversifiable risk of an asset with respect to the market portfolio.

The example will demonstrate how to **calculate this Beta** in CAPM equation.
Click here to see tab *CAPM* in the shared Excel file online.

ORNSTEIN–UHLENBECK PROCESS

This is a type of Stochastic Differential Equation. Its SDE is given by:

$$dX_t = -\kappa X_t dt + \sigma dW_t, X_0 = x$$

with $\sigma, k > 0$ being real constants. This stochastic differential equation has a solution known as the **Ornstein-Uhlenbeck process**.

$$dX_t = \kappa (\theta - X_t) dt + \sigma dW_t, X_0 = x$$

which is the mean-reverting process, which is drawn to the value of θ.
Click here to see tab *Ornstein–Uhlenbeck Process* in the shared Excel file online.

OPTION GREEKS

The Greeks are a group of mathematical derivatives that can be used to help us manage or understand the risks in our portfolio. Different types of greeks are:

- **Delta** tells the rate of change in price of derivative due to change in price of underlying asset.

  When we consider delta hedging, we add the deltas for the individual assets and derivatives (taking into account, of course, the number of units held for each). If this sum is zero and if the underlying asset prices follow a diffusion, then the portfolio is instantaneously risk-free.

- **Gamma** is the rate of change of Delta with the price of the underlying asset.
- **Vega** is the rate of change of the price of the derivative with respect to a change in the assumed level of volatility of St.
- **Rho** tells us about the sensitivity of the derivative price to changes in the risk-free rate of interest.
- Lambda is the sensitivity of the derivative price to changes in the dividend yield on the underlying asset.
- **Theta** tells the rate of change in the price of derivative due to change in the duration.

Click here to see tab *Option Greeks* in the shared Excel file online.

BINOMIAL MODEL

We will consider a model for stock prices in discrete time. We will use S_t to represent the price of a non-dividend-paying stock at discrete time intervals t ($t = 0, 1, 2, ...$). For $t > 0$, S_t is random.

At all points in time there are no constraints (positive or negative) on how much we can hold in stock or cash.

At time t there are 2t possible states $(t, 1)$, $(t, 2)$, ..., $(t, 2t)$. In state (t, j), the price of the underlying stock is S_t(j). From this state the price can

- go up to $S_{t+1}(2j-1) = S_t(j)u_t(j)$ and State $(t+1, 2j-1)$;
- or down to $S_{t+1}(2j) = S_t(j)d_t(j)$ and State $(t+1, 2j)$

This model allows share prices to move in a way which is clearly much simpler. The binomial model is recognized as an effective model (provided that the time to maturity is broken up into a suitable number of subperiods) for pricing and valuing derivative contracts.

Replicating portfolio replicates, precisely, the payoff at time 1 on the derivative without any risk. It is also a simple example of a hedging strategy: that is, an investment strategy that reduces the amount of risk carried by the issuer of the contract.

There is a **state price deflator approach** in the **n-period binomial tree**.

$$A_n = \exp^{-rn} \quad \left(\frac{q}{p}\right)^{N_n} \left(\frac{1-q}{1-p}\right)^{n-N_n}$$

where N_n is the number of steps up to time n. The fair price for the derivative is then:

$$V_0 = E_p\left[A_n V_n\right]$$

Click here to see tab *Binomial Tree (1)* in the shared Excel file online.
Click here to see tab *Binomial Tree (2)* in the shared Excel file online.

BLACK SCHOLES MODEL

Black-Scholes Model is the most commonly used model for option pricing under which share prices evolve in continuous time and are characterised at any point in time by a continuous distribution rather than a discrete distribution. Assumptions underlying this model are:

- The price of the underlying share follows a geometric Brownian motion.
- There are no risk-free arbitrage opportunities.
- The risk-free rate of interest is constant, the same for all maturities, and the same for borrowing or lending.
- Unlimited short selling (that is, negative holdings) is allowed
- There are no taxes or transaction costs.
- The underlying asset can be traded continuously and in infinitesimally small numbers of units.

The example will show the application of this model.
Click here to see tab *Black Scholes Option Pricing* in the shared Excel file online.

CREDIT RISK

Credit risk exists when a party may default on its obligations. Credit risk is calculated as an expected loss:

$$\text{Expected Loss} = \text{Exposure} \times \text{Probability of Default} \times \text{Loss Given Default}$$

The **Merton model** states that the value of a firm's equity is a call option on the value of the firm and the liability is treated as a strike price. This optionality can then be valued using the **Black-Scholes option pricing formula**. The examples will show the application of Credit Risk.
Click here to see tab *Credit Risk (1)* in the shared Excel file online.
Click here to see tab *Credit Risk (2)* in the shared Excel file online.

RUIN THEORY

Suppose that at time 0 the insurer has an amount of money set aside for this portfolio. This amount of money is called the initial surplus and is denoted U. It will always be assumed that $U \geq 0$. The insurer needs this initial surplus because future premium income on its own may not be sufficient to cover future claims. Here we are ignoring expenses.

The insurer's surplus falls below zero as a result of the claim at some time T. When the surplus falls below zero, the insurer has run out of money, and it is said that ruin or insolvency has occurred. Another way of looking at the probability of ruin is to think of it as the probability that, at some future time, the insurance company will need to provide more capital to finance this particular portfolio.

The following two probabilities are defined:

$$\psi(Y) = P[Y(t) < 0, \text{ for some } t,\ 0 < t < \infty]$$
$$\psi(Y, \tau) = P[Y(\tau) < 0, \text{ for some } t,\ 0 < \tau \leq t].$$

where $\psi(U)$ is the probability of ultimate ruin (given initial surplus U) and $\psi(Y, \tau)$ is the probability of ruin within time t (given initial surplus U).

Lundberg's inequality states that:

$$\psi(U) = \exp\{-RU\}$$

where U is the insurer's initial surplus and $\psi(U)$ is the probability of ultimate ruin. R is a parameter associated with a surplus process known as the adjustment coefficient. Its value depends on the distribution of aggregate claims and on the rate of premium income.

R can be interpreted as measuring risk. The larger the value of R, the smaller the upper bound for $\psi(U)$ will be.

We will see the application of these concepts in some examples.
Click here to see tab *Ruin Theory (1)* in the shared Excel file online.
Click here to see tab *Ruin Theory (2)* in the shared Excel file online.
Click here to see tab *Ruin Theory (3)* in the shared Excel file online.

RUN-OFF TRIANGLES

Run-off triangles (delay triangles) usually arise in types of insurance (particularly nonlife insurance), where it may take some time after a loss until the full extent of the claims which have to be paid are known. It is important that the claims are attributed to the year in which the policy was written.

The insurance company needs to know how much it is liable to pay in claims so it can calculate how much surplus it has made. However, it may be many years before it knows the exact claim totals. There are many causes for delays in the finalizing of the claim totals. The delay may occur before notification of the claim and/or between notification and final settlement.

Although the insurer does not know the exact figure for the total claims each year, it attempts to estimate that figure with as much confidence and accuracy as possible.

The claims data are presented in a format that is by accident year and development year.

The year in which the incident occurred and the insurer was at risk is called accident year. The number of years until a payment is made is called the delay, or development period.

The four standard methods for projecting run-off triangles are:

- Basic chain ladder method
- Inflation-adjusted chain ladder method
- Average cost per claim method
- Bornhuetter-Ferguson method

We will see the application of these methods.
Click here to see tab *Run-Off Triangles (1)* in the shared Excel file online.
Click here to see tab *Run-Off Triangles (2)* in the shared Excel file online.