FAM-S

A. REVIEW OF PROBABILITY

A1. BASIC PROBABILITY

Cumulative Distribution Function: $F(x) = \Pr(X \le x)$

Survival Function: $S(x) = 1 - F(x) = \Pr(X > x)$

Probability Density Function: $f(x) = \frac{d}{dx} F(x) = -\frac{d}{dx} S(x)$

Hazard Rate Function: $\lambda(x) = \frac{f(x)}{S(x)} = -\frac{d}{dx} \log S(x) \rightarrow S(x) = e^{-\int_{-\infty}^{x} \lambda(t) dt}$

Cumulative Hazard Function: $\Lambda(x) = \int_{-\infty}^{x} \lambda(t)\, dt \rightarrow S(x) = e^{-\Lambda(x)}$

Transformation: $Y = g(X) \rightarrow f_Y(y) = f_X\left(g^{-1}(y)\right) \left| \frac{dg^{-1}(y)}{dy} \right|$

A2. EXPECTATION & VARIANCE

Expectation (Discrete X): $E[X] = \sum x \Pr(X = x) \rightarrow E[g(X)] = \sum g(x) \Pr(X = x)$

Expectation (Continuous X): $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx \rightarrow E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

Variance: $Var(X) = E\left[X^2\right] - E[X]^2$

Standard Deviation: $SD(X) = \sqrt{Var(X)}$

Covariance: $Cov(X, Y) = E[XY] - E[X]\, E[Y]$

Correlation Coefficient: $Corr(X, Y) = \dfrac{Cov(X, Y)}{SD(X)\ SD(Y)}$

Linear Combination: $W = aX + bY$
$\rightarrow E[W] = aE[X] + bE[Y]$
$\rightarrow Var(W) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X, Y)$

Sum of iid X: $S = X_1 + \cdots + X_n$
$\rightarrow E[S] = nE[X]$
$\rightarrow Var(S) = nVar(X)$

A3. CONDITIONAL PROBABILITY

Conditional Probability: $\Pr(X=x|Y=y)=\frac{\Pr(X=x,\ Y=y)}{\Pr(Y=y)}=\frac{\Pr(X=x)\Pr(Y=y|X=x)}{\Pr(Y=y)}$

Conditional Density Function: $f(x|y)=\frac{f(x,y)}{f(y)}=\frac{f(x)f(y|x)}{f(y)}$

Conditional expectation (Discrete): $E[X|Y=y]=\sum x\Pr(X=x|Y=y)$

Conditional expectation (Continuous): $E[X|Y=y]=\int_{-\infty}^{\infty}xf(x|y)dx$

Law of Total Probability: $\Pr(X\le x)=E[\Pr(X\le x|Y)]$

Law of Total Expectation: $E[X]=E[E[X|Y]]$

Law of Total Variance: $Var(X)=E[Var(X|Y)]+Var(E[X|Y])$

B. SEVERITY, FREQUENCY & AGGREGATE MODELS

B1. SEVERITY DISTRIBUTIONS

Distribution	Probability density function	Formulas worth memorizing		
Uniform	$f(x)=\frac{1}{b-a},\ a\le x\le b$	$F(x)=\frac{x-a}{b-a}$	$E[X]=\frac{a+b}{2}$	$Var(X)=\frac{(b-a)^2}{12}$
Exponential	$f(x)=\frac{1}{\theta}e^{-\frac{x}{\theta}},\ x>0$	$F(x)=1-e^{-\frac{x}{\theta}}$	$E[X]=\theta$	$Var(X)=\theta^2$
Weibull	$f(x)=\frac{\tau}{x}\left(\frac{x}{\theta}\right)^{\tau}e^{-\left(\frac{x}{\theta}\right)^{\tau}},\ x>0$	$F(x)=1-e^{-\left(\frac{x}{\theta}\right)^{\tau}}$		
Gamma	$f(x)=\frac{\left(\frac{1}{\theta}\right)^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}e^{-\frac{x}{\theta}},\ x>0$	$F(x)=\Pr(X^*\ge\alpha)$ X^* is Poisson with $\lambda=\frac{x}{\theta}$ If α is an integer.	$E[X]=\alpha\theta$	$Var(X)=\alpha\theta^2$
Beta	$f(x)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$, $0<x<1$		$E[X]=\frac{a}{a+b}$ If a and b are integers.	$E[X^2]=\frac{a(a+1)}{(a+b)(a+b+1)}$ If a and b are integers.
Pareto	$f(x)=\frac{\alpha\theta^{\alpha}}{(x+\theta)^{\alpha+1}},\ x>0$	$F(x)=1-\left(\frac{\theta}{x+\theta}\right)^{\alpha}$	$E[X]=\frac{\theta}{\alpha-1}$ If $\alpha>1$ is an integer.	
Single P. Pareto	$f(x)=\frac{\alpha\theta^{\alpha}}{x^{\alpha+1}},\ x>\theta$	$F(x)=1-\left(\frac{\theta}{x}\right)^{\alpha}$	$E[X]=\frac{\alpha\theta}{\alpha-1}$	
Lognormal	$f(x)=\frac{1}{x\sigma\sqrt{2\pi}}e^{-\frac{(\log x-\mu)^2}{2\sigma^2}},\ x>0$	$F(x)=N\left(\frac{\log x-\mu}{\sigma}\right)$	$E[X]=e^{\mu+\frac{\sigma^2}{2}}$	$E[X^2]=e^{2\mu+2\sigma^2}$

B2. FREQUENCY DISTRIBUTIONS

Distribution	Probability mass function	Formulas worth memorizing	
Poisson	$P(x) = \frac{e^{-\lambda}\lambda^x}{x!}, x = 0, 1, 2, \ldots, \infty$	$E[N] = \lambda$	$Var(N) = \lambda$
Binomial	$P(x) = \binom{m}{x} q^x (1-q)^{m-x}, x = 0, 1, 2, \ldots, m$	$E[N] = mq$	$Var(N) = mq(1-q)$
Geometric	$P(x) = \left(\frac{1}{1+\beta}\right)\left(\frac{\beta}{1+\beta}\right)^x, x = 0, 1, 2, \ldots, \infty$	$E[N] = \beta$	$Var(N) = \beta(1+\beta)$
Negative Binomial	$P(x) = \binom{x+r-1}{x}\left(\frac{1}{1+\beta}\right)^r\left(\frac{\beta}{1+\beta}\right)^x, x = 0, 1, 2, \ldots, \infty$	$E[N] = r\beta$	$Var(N) = r\beta(1+\beta)$

Zero-truncated distributions.: $p_x^T = \frac{p_x}{1-p_0}$

Zero-modified distributions.: $p_x^M = \left(1 - p_0^M\right)\left(\frac{p_x}{1-p_0}\right)$

B3. AGGREGATE MODELS

Individual risk model: $S = X_1 + X_2 + \cdots + X_n$ $\rightarrow$ $E[S] = nE[X]$

$\rightarrow$ $Var(S) = nVar(X)$

Collective risk model: $S = X_1 + X_2 + \cdots + X_N$ $\rightarrow$ $E[S] = E[N]E[X]$

$\rightarrow$ $Var(S) = E[N]Var(X) + Var(N)E[X]^2$

Normal Distribution: $X_i \overset{iid}{\sim} N(\mu,\ \sigma^2)$ $\rightarrow$ $S^L = X_1 + X_2 + \cdots + X_n \sim N(n\mu,\ n\sigma^2)$

Exponential Distribution: $X_i \overset{iid}{\sim} Exp(\theta)$ $\rightarrow$ $S^L = X_1 + X_2 + \cdots + X_n \sim Gamma(\alpha = n,\ \theta)$

Normal approximation: $S \dot{\sim} N\left(\mu = E[S], \sigma^2 = Var(S)\right)$ $\rightarrow$ $Pr(S \le k) \approx N\left(\frac{k-\mu}{\sigma}\right)$

B4. MEASURES OF RISKS

Coherent risk measures:

1. **Translation Invariance:** $\rho(X + c) = \rho(X) + c$
2. **Positive Homogeneity:** $\rho(cX) = c\rho(X)$
3. **Subadditivity:** $\rho(X + Y) \le \rho(X) + \rho(Y)$
4. **Monotonicity:** $\rho(X) \le \rho(Y)$ If $\Pr(X \le Y) = 1$

Value-at-Risk: $\text{VaR}_\alpha(X)$ Where $\Pr(X \le \text{VaR}_\alpha(X)) = \alpha$.

Tail-Value-at-Risk: $\text{TVAR}_\alpha(X) = E[X|X > \text{VaR}_\alpha(X)] = \text{VaR}_\alpha(X) + \frac{E[X] - E[X \wedge \text{VaR}_\alpha(X)]}{1-\alpha}$

Equilibrium Distribution: $f_e(x) = \frac{S(x)}{E[X]},\ X > 0$ $\rightarrow$ $E[X_e] = \frac{E[X^2]}{2E[X]}$

C. COVERAGE MODIFICATIONS

C1. PAYMENT PER LOSS

Policy	Payment per loss	Expected payment per loss
With ordinary deductible d	$Y^L = \begin{cases} 0, & X < d \\ X - d, & X \geq d \end{cases}$	$E\left[Y^L\right] = E[X] - E[X \wedge d]$
With franchise deductible d^*	$Y^L = \begin{cases} 0, & X \leq d^* \\ X, & X > d^* \end{cases}$	$E\left[Y^L\right] = E\left[X \mid X > d^*\right]$
With maximum covered loss u	$Y^L = \begin{cases} X, & X \leq u \\ u, & X > u \end{cases}$	$E\left[Y^L\right] = E[X \wedge u]$
With d and u	$Y^L = \begin{cases} 0, & X \leq d \\ X - d, & d < X \leq u \\ u - d, & X > u \end{cases}$	$E\left[Y^L\right] = E[X \wedge u] - E[X \wedge d]$
With $d,\ u$ and coinsurance factor α	$Y^L = \begin{cases} 0, & X \leq d \\ \alpha(X - d), & d < X \leq u \\ \alpha(u - d), & X > u \end{cases}$	$E\left[Y^L\right] = \alpha(E[X \wedge u] - E[X \wedge d])$
With d, u, α and inflation rate r	$Y^L = \begin{cases} 0, & X \leq \frac{d}{1+r} \\ \alpha(1+r)\left(X - \frac{d}{1+r}\right), & \frac{d}{1+r} < X \leq \frac{u}{1+r} \\ \alpha(1+r)\left(\frac{u}{1+r} - \frac{d}{1+r}\right), & X > \frac{u}{1+r} \end{cases}$	$E\left[Y^L\right] = \alpha(1+r)\left(E\left[X \wedge \frac{u}{1+r}\right] - E\left[X \wedge \frac{d}{1+r}\right]\right)$

Loss elimination ratio: $\text{LER} = 1 - \frac{E\left[Y^L\right]}{E[X]}$

C2. PAYMENT PER PAYMENT

Payment per Payment $Y^P = Y^L | X > d \quad \rightarrow \quad E\left[Y^P\right] = \frac{E\left[Y^L\right]}{\Pr(X > d)}$

Policy	Loss	Payment per payment
With ordinary deductible d	$X \sim Unif(0, b)$	$Y^P \sim Unif(0, b - d)$
	$X \sim Exp(\theta)$	$Y^P \sim Exp(\theta)$
	$X \sim Pareto(\alpha, \theta)$	$Y^P \sim Pareto(\alpha, \theta + d)$

C3. REINSURANCE

Proportional reinsurance	Insurer pays	Reinsurer pays
With quota share α	$Y=(1-\alpha)X$	$Y=\alpha X$
With retention u and surplus share α	$Y=\begin{cases}X, & X\le u\\ u+(1-\alpha)(X-u), & X>u\end{cases}$	$Y=\begin{cases}0, & X\le u\\ \alpha(X-u), & X>u\end{cases}$

Excess of loss reinsurance	Insurer pays	Reinsurer pays
Covers losses above d	$Y=\begin{cases}X, & X\le d\\ d, & X>d\end{cases}$	$Y=\begin{cases}0, & X\le d\\ X-d, & X>d\end{cases}$
Covers losses above d but below u	$Y=\begin{cases}X, & X\le d\\ d, & d\le X<u\\ X+d-u, & X>u\end{cases}$	$Y=\begin{cases}0, & X\le d\\ X-d, & d\le X<u\\ u-d, & X>u\end{cases}$

D. MAXIMUM LIKELIHOOD ESTIMATION

D1. MLE WITH COMPLETE DATA

For distributions that belong to the exponential family:

1. Determine $L(\theta)$.
2. Apply natural logarithm, obtain $l(\theta)=\log L(\theta)$.
3. Take the first derivative with respect to the parameter, obtain $l'(\theta)$.
4. Set $l'(\theta)=0$, obtain $\hat{\theta}$, which is the MLE.

Distribution	Likelihood Function	Maximum likelihood estimate(s)	
Exponential	$L(\theta)=f(x_1)\ldots f(x_n)$	$\hat{\theta}=\bar{x}$	
Normal		$\hat{\mu}=\bar{x}$	$\hat{\sigma}^2=\frac{1}{n}\sum(x_i-\hat{\mu})^2$
Lognormal		$\hat{\mu}=\frac{1}{n}\sum\log x_i$	$\hat{\sigma}^2=\frac{1}{n}\sum(\log x_i-\hat{\mu})^2$
Uniform		$\hat{b}=\max(x_1,\ldots,x_n)$	
Binomial	$L(\theta)=p(x_1)\ldots p(x_n)$	$\hat{q}=\frac{\bar{x}}{m}$	
Poisson		$\hat{\lambda}=\bar{x}$	

D2. MLE WITH INCOMPLETE DATA

	Likelihood function	Note
Grouped data	$L = (F(c_1) - F(c_0))^{m_1} \dots (F(c_n) - F(c_{n-1}))^{m_n}$	Where $c_0 < c_1 < \dots < c_n$ are interval boundaries.
Left-truncated data	$L = \frac{f(x_1)}{S(d)} \dots \frac{f(x_n)}{S(d)}$	Losses below d are not reported.
Right-censored data	$L = f(x_1) \dots f(x_n) S(u)^m$	Losses are capped at u.
Left-truncated & Right-censored data	$L = \frac{f(x_1)}{S(d)} \dots \frac{f(x_n)}{S(d)} \left(\frac{S(u)}{S(d)}\right)^m$	Losses below d are not reported. Losses are capped at u.

E. CLASSICAL CREDIBILITY

E1. FULL CREDIBILITY

We want...to be within k of the mean p of the time.	Range parameter, k Note: $z = N^{-1}\left(\frac{1+p}{2}\right)$	Total number of exposures needed, e_F	Total number of claims needed, n_F
Average number of claims	$k = \frac{z\sqrt{\frac{\mathrm{Var}(N)}{e_F}}}{E[N]}$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(N)}{E[N]^2}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(N)}{E[N]^2} \times E[N]$
Total number of claims	$k = \frac{z\sqrt{e_F \mathrm{Var}(N)}}{e_F E[N]}$		
Average claim size	$k = \frac{z\sqrt{\frac{\mathrm{Var}(X)}{n_F}}}{E[X]}$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(X)}{E[X]^2} \times \frac{1}{E[N]}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(X)}{E[X]^2}$
Average aggregate claims	$k = \frac{z\sqrt{\frac{\mathrm{Var}(S)}{e_F}}}{E[S]}$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(S)}{E[S]^2}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(S)}{E[S]^2} \times E[N]$
Total aggregate claims	$k = \frac{z\sqrt{e_F \mathrm{Var}(S)}}{e_F E[S]}$		

E2. POISSON FREQUENCY

We want...to be within k of the mean p of the time.	Frequency	Total number of exposures needed, e_F	Total number of claims needed, n_F
Average number of claims	$N \sim Poisson(\lambda)$	$e_F = \left(\frac{z}{k}\right)^2 \times \frac{1}{\lambda}$	$n_F = \left(\frac{z}{k}\right)^2$
Total number of claims			
Average claim size		$e_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(X)}{E[X]^2} \times \frac{1}{\lambda}$	$n_F = \left(\frac{z}{k}\right)^2 \times \frac{\mathrm{Var}(X)}{E[X]^2}$
Average aggregate claims		$e_F = \left(\frac{z}{k}\right)^2 \times \left(1 + \frac{\mathrm{Var}(X)}{E[X]^2}\right) \times \frac{1}{\lambda}$	$n_F = \left(\frac{z}{k}\right)^2 \times \left(1 + \frac{\mathrm{Var}(X)}{E[X]^2}\right)$
Total aggregate claims			

E3. PARTIAL CREDIBILITY

Credibility factor: $Z = \sqrt{\dfrac{\text{Number of exposures available}}{\text{Number of exposures needed for full credibility}}}$

or $Z = \sqrt{\dfrac{\text{Number of claims available}}{\text{Number of claims needed for full credibility}}}$

Credibility premium: $P = Z \times \text{Observation} + (1 - Z) \times \text{Manual Rate}$

F. RATEMAKING & LOSS RESERVING

F1. RATEMAKING DATA

Exposure	Written exposure Earned exposure Unearned exposure
Premium	Written premium Earned premium Unearned premium
Claim	Reported claims Unreported claims
Loss	Reported loss = Paid loss + Case reserve IBNR reserve = Incurred but not reported reserve IBNER reserve = Incurred but not enough reported reserve Ultimate loss = Reported loss + IBNR reserve + IBNER reserve
Expenses	ALAE = Allocated loss adjustment expenses ULAE = Unallocated loss adjustment expenses

Basic formulas:

$$\text{Frequency} = \frac{\text{Number of Claims}}{\text{Number of Exposures}}$$

$$\text{Severity} = \frac{\text{Losses}}{\text{Number of Claims}}$$

Pure Premium/Loss Cost $= \dfrac{\text{Losses}}{\text{Number of Exposures}} = \text{Frequency} \times \text{Severity}$

Loss Ratio $= \dfrac{\text{Losses}}{\text{Premium}}$

Expense Ratio $= \dfrac{\text{Expenses}}{\text{Premium}}$

F2. AGGREGATION METHODS

Aggregation methods	Premium/Exposure	Loss
Calendar year	Transaction date	Transaction date
Calendar-accident year	Transaction date	Accident date
Policy year	Effective date	Effective date

F3. LOSS RESERVING METHODS

Expected loss ratio method	1. Ultimate losses = Earned Premium × Expected Loss Ratio 2. Loss Reserve = Ultimate Losses − Paid Losses
Chain-ladder method	1. Prepare a run-off triangle for paid losses. 2. Calculate age-to-age factors using average factor method or mean factor method. 3. Calculate age-to-ultimate factor f_{ULT}, which is the product of age-to-age factors. 4. Ultimate Losses = Paid Losses × f_{ULT} 5. Loss reserve = Ultimate Losses − Paid Losses
Bornhuetter-Ferguson method	1. Prepare a run-off triangle for paid losses. 2. Calculate age-to-age factors using average factor method or mean factor method. 3. Calculate age-to-ultimate factor f_{ULT}, which is the product of age-to-age factors. 4. Ultimate Losses = Paid Losses + Earned Premium × Expected Loss Ratio × $\left(1 - \frac{1}{f_{ULT}}\right)$ 5. Loss reserve = Ultimate Losses − Paid Losses

F4. PRICING FORMULA

General formula: Premium = Losses + Loss adjustment expenses + Fixed Expenses + Variable Expenses + Profit

Premium: $P = L + LAE + F + (V + Q)P \quad \rightarrow \quad P = \dfrac{L + LAE + F}{1 - V - Q}$

Permissible loss ratio: $R = 1 - V - Q \quad \rightarrow \quad P = \dfrac{L + LAE + F}{R}$

Adjustments to data: Premium at current rates = Earned premium × $\dfrac{\text{Current rate level}}{\text{Historical average rate level}}$

Ultimate losses = Reported losses × Development factor

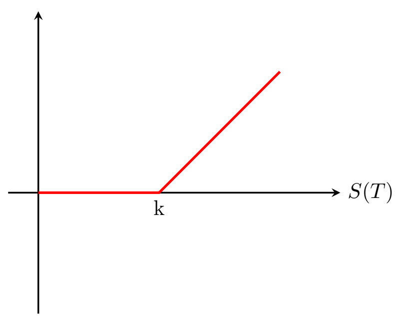
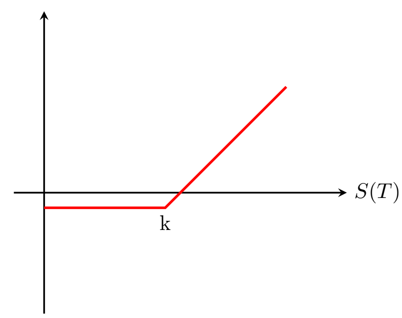
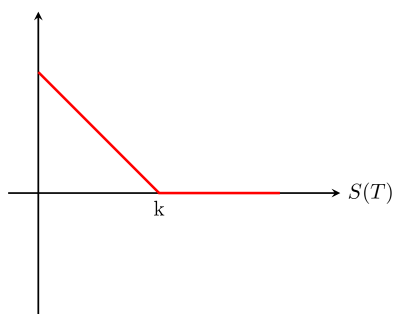
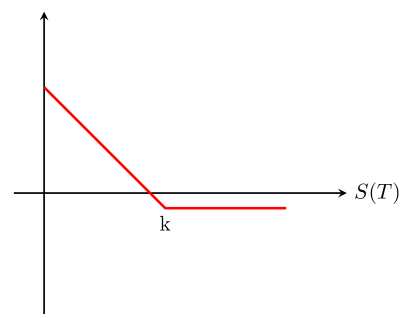
Trended losses = Reported losses × Trend factor

F5. RATEMAKING METHODS

Loss cost method	$\text{Projected loss cost including LAE} = \dfrac{\text{Trended and ultimate losses and LAE}}{\text{Number of earned exposures}}$ $\text{Indicated rate} = \dfrac{\text{Projected loss cost+Fixed expenses per exposure}}{\text{Permissible loss ratio}}$ $\text{Indicated rate change} = \dfrac{\text{Indicated rate}}{\text{Current rate}} - 1$
Loss ratio method	$\text{Projected loss and LAE ratio} = \dfrac{\text{Trended and ultimate losses and LAE}}{\text{Earned premiums at current rate level}}$ $\text{Indicated rate change} = \dfrac{\text{Projected loss and LAE ratio + Fixed expense ratio}}{\text{Permissible loss ratio}} - 1$ $\text{Indicated rate} = \dfrac{\text{Earned premiums at current rate level}}{\text{Number of earned exposures}} \times (1 + \text{Indicated rate change})$

G. Option Pricing

G1. PUT AND CALL OPTIONS

Financial derivative	**Payoff**	**Profit**
Call option	$C(T) = \max(0, S(T) - K)$ 	$\text{Profit} = C(T) - C(0)e^{rT}$ 
Put option	$P(T) = \max(0, K - S(T))$ 	$\text{Profit} = P(T) - P(0)e^{rT}$ 

Put-call parity: $C(0) - P(0) = S(0) - Ke^{-rT}$

G2. BINOMIAL OPTION PRICING MODEL

Stock price at time h: $S_u = S(0) \times u$ or $S_d = S(0) \times d$

Option payoff at time h: V_u or V_d

Replicating portfolio: $\Delta = \dfrac{V_u - V_d}{S_u - S_d}$ & $B = e^{-rh}\left(\dfrac{uV_u - dV_u}{u-d}\right)$

Option price: $V(0) = \Delta S(0) + B$

Risk neutral probability: $p^* = \dfrac{e^{rh} - d}{u - d}$

Option price: $V(0) = e^{-rh}\left(p^* V_u + (1-p^*) V_d\right)$

G3. BLACK-SCHOLES-MERTON MODEL

Stock price at time T: $S(T) = S(0)e^{\left(r - \frac{\sigma^2}{2}\right)T + \sigma\sqrt{T}Z}$ where $Z \sim N(0,1)$

Call price: $C(0) = S(0)N(d_1) - Ke^{-rT}N(d_2)$

Call delta: $\Delta_C = N(d_1)$

Put price: $P(0) = Ke^{-rT}N(-d_2) - SN(-d_1)$

Put delta: $\Delta_p = -N(-d_1)$

Where: $d_1 = \dfrac{\log \frac{S(0)}{K} + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}$ & $d_2 = d_1 - \sigma\sqrt{T}$

FAM-L

A. REVIEW OF PROBABILITY

A1. BASIC PROBABILITY

Cumulative Distribution Function: $F(x) = \Pr(X \le x)$

Survival Function: $S(x) = 1 - F(x) = \Pr(X > x)$

Probability Density Function: $f(x) = \frac{d}{dx} F(x) = -\frac{d}{dx} S(x)$

Hazard Rate Function: $\lambda(x) = \frac{f(x)}{S(x)} = -\frac{d}{dx} \log S(x) \quad \rightarrow \quad S(x) = e^{-\int_{-\infty}^{x} \lambda(t) dt}$

Cumulative Hazard Function: $\Lambda(x) = \int_{-\infty}^{x} \lambda(t)\, dt \quad \rightarrow \quad S(x) = e^{-\Lambda(x)}$

A2. EXPECTATION & VARIANCE

Expectation (Discrete X): $E[X] = \sum x \Pr(X = x) \quad \rightarrow \quad E[g(X)] = \sum g(x) \Pr(X = x)$

Expectation (Continuous X): $E[X] = \int_{-\infty}^{\infty} x f(x)\, dx \quad \rightarrow \quad E[g(X)] = \int_{-\infty}^{\infty} g(x) f(x)\, dx$

Variance: $Var(X) = E\left[X^2\right] - E[X]^2$

Standard Deviation: $SD(X) = \sqrt{Var(X)}$

Covariance: $Cov(X, Y) = E[XY] - E[X]E[Y]$

Correlation Coefficient: $Corr(X, Y) = \dfrac{Cov(X, Y)}{SD(X)\ SD(Y)}$

Linear Combination: $W = aX + bY$

$E[W] = aE[X] + bE[Y]$

$Var(W) = a^2 Var(X) + b^2 Var(Y) + 2ab\, Cov(X, Y)$

Sum of iid X: $S = X_1 + \cdots + X_n$

$E[S] = E[X_1 + \cdots + X_n] = nE[X]$

$Var(S) = Var(X_1 + \cdots + X_n) = nVar(X)$

A3. CONDITIONAL PROBABILITY

Conditional expectation (Discrete): $E[X \mid Y=y]=\sum x \operatorname{Pr}(X=x \mid Y=y)$

Conditional expectation (Continuous): $E[X \mid Y=y]=\int_{-\infty}^{\infty} x f(x \mid y) d x$

Law of total expectation: $E[X]=E[E[X|Y]]$

Law of Total Variance: $Var(X)=E[Var(X|Y)]+Var(E[X|Y])$

A4. COMMON DISTRIBUTIONS

Uniform distribution: $f_X(x)=\frac{1}{b-a}$ for $a \leq x \leq b$ $\rightarrow$ $E[X]=\frac{a+b}{2}$ $\quad Var(X)=\frac{(b-a)^2}{12}$

Exponential distribution: $f_X(x)=\frac{1}{\theta} e^{-\frac{x}{\theta}}$ for $x>0$ $\rightarrow$ $E[X]=\theta$ $\quad Var(X)=\theta^2$

Normal approximation: $S \dot{\sim}\left(\mu=E[S],\ \sigma^2=Var(S)\right)$ $\rightarrow$ $\operatorname{Pr}(S \leq k) \approx N\left(\frac{k-\mu}{\sigma}\right)$

B. REVIEW OF FINANCIAL MATHEMATICS

B1. INTEREST RATES

Annual effective interest rate: i

Rate of discount: $d=\frac{i}{1+i}=iv=1-v$

Discounting rate: $v=\frac{1}{1+i}=1-d$

Continuously compounded interest rate: $\delta=\log(1+i)$

Nominal interest rate: $i^{(m)}=m\left((1+i)^{\frac{1}{m}}-1\right)$

Nominal rate of discount: $d^{(m)}=m\left(1-(1-d)^{\frac{1}{m}}\right)$

B2. PRESENT VALUES

PV of n-year certain annuity-due: $\ddot{a}_{\overline{n}|}=\frac{1-v^n}{d}$ (1 at the beginning of each year)

PV of n-year certain annuity-immediate: $a_{\overline{n}|}=\frac{1-v^n}{i}$ (1 at the end of each year)

PV of n-year continuous certain annuity: $\bar{a}_{\overline{n}|}=\frac{1-v^n}{\delta}$ (1 per year continuously)

PV of n-year 1/m-thly certain annuity-due: $\ddot{a}_{\overline{n}|}^{(m)}=\frac{1-v^n}{d^{(m)}}$ ($1/m$ at the beginning of each of the nm periods)

C. SURVIVAL MODELS

C1. SURVIVAL FUNCTION

Complete future lifetime: T_x $\rightarrow$ The pdf of T_x is $f_x(t) = {}_tp_x \ \mu_{x+t}$.

Curtate future lifetime: $K_x = \lfloor T_x \rfloor$ $\rightarrow$ The pmf of K_x is $\Pr(K_x = k) = {}_kp_x \ q_{x+k}$.

Survival function: $S_x(t) = \Pr(T_x > t)$

Formulas:

$S_x(u+t) = S_x(u) \ S_{x+u}(t)$ $\rightarrow$ $S_{x+u}(t) = \dfrac{S_x(u+t)}{S_x(u)}$

$S_0(x+t) = S_0(x) \ S_x(t)$ $\rightarrow$ $S_x(t) = \dfrac{S_0(x+t)}{S_0(x)}$

Three conditions:

(1) $S_x(0) = 1$

(2) $\lim_{t \to \infty} S_x(t) = 0$

(3) $S_x(t)$ must be a non-increasing function of t.

C2. ACTUARIAL NOTATION

Survival probability: ${}_tp_x = \Pr(T_x > t)$

Mortality probability: ${}_tq_x = 1 - {}_tp_x = \Pr(T_x \le t)$

Formulas:

${}_{t+u}p_x = {}_tp_x \ {}_up_{x+t}$

${}_{t+u}q_x = {}_tq_x + {}_tp_x \ {}_uq_{x+t}$

${}_{t|u}q_x = {}_tp_x \ {}_uq_{x+t} = {}_tp_x - {}_{t+u}p_x = {}_{t+u}q_x - {}_tq_x$

C3. LIFE TABLES

Number of lives: l_x

Number of deaths: ${}_td_x = l_x - l_{x+t}$

Formulas:

${}_tp_x = \dfrac{l_{x+t}}{l_x}$

${}_tq_x = \dfrac{{}_td_x}{l_x} = \dfrac{l_x - l_{x+t}}{l_x}$

${}_{t|u}q_x = \dfrac{{}_ud_{x+t}}{l_x} = \dfrac{l_{x+t} - l_{x+t+u}}{l_x}$

C4. FORCE OF MORTALITY

Definition: $\mu_x(t) = \mu_{x+t} = \frac{f_x(t)}{S_x(t)} = -\frac{d}{dt}\log S_x(t)$

Condition: $\lim_{t\to\infty} \int_0^t \mu_s ds = \infty$

Formulas: ${}_tp_x = e^{-\int_0^t \mu_{x+s}ds} = e^{-\int_x^{x+t} \mu_s ds}$

${}_tq_x = \int_0^t {}_sp_x\ \mu_{x+s}ds$

${}_{t|u}q_x = \int_t^{t+u} {}_sp_x\ \mu_{x+s}ds$

C5. EXPECTED FUTURE LIFETIME

Expectations: $E[T_x] = \overset{o}{e}_x = \int_0^\infty t\ {}_tp_x\ \mu_{x+t}dt = \int_0^\infty {}_tp_x dt$

$E[K_x] = e_x = \sum_{k=1}^{\infty} k\ {}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty} {}_kp_x$

$E[\min(T_x, n)] = \overset{o}{e}_{x:\overline{n|}} = \int_0^n t\ {}_tp_x\ \mu_{x+t}dt + n\,{}_np_x = \int_0^n {}_tp_x dt$

$E[\min(K_x, n)] = e_{x:\overline{n|}} = \sum_{k=1}^{n-1} k\ {}_kp_x\ q_{x+k} + n\,{}_np_x = \sum_{k=1}^{n} {}_kp_x$

Second moments: $E[T_x^2] = \int_0^\infty t^2\,{}_tp_x\ \mu_{x+t}dt = \int_0^\infty 2t\,{}_tp_x dt$

$E[K_x^2] = \sum_{k=1}^{\infty} k^2\,{}_kp_x\ q_{x+k} = \sum_{k=1}^{\infty} (2k-1)\,{}_kp_x = 2\sum_{k=1}^{\infty} k\ {}_kp_x - e_x$

$E\left[\min(T_x, n)^2\right] = \int_0^n t^2\,{}_tp_x\ \mu_{x+t}dt + n^2\,{}_np_x = \int_0^n 2t\,{}_tp_x dt$

$E\left[\min(K_x, n)^2\right] = \sum_{k=1}^{n-1} k^2\,{}_kp_x\ q_{x+k} + n^2\,{}_np_x = \sum_{k=1}^{n} (2k-1)\,{}_kp_x$

Variance: $Var(T_x) = E[T_x^2] - E[T_x]^2$

$Var(K_x) = E[K_x^2] - E[K_x]^2$

Recursive formulas: $\overset{o}{e}_x = \overset{o}{e}_{x:\overline{n|}} + {}_np_x\ \overset{o}{e}_{x+n}$

$e_x = e_{x:\overline{n|}} + {}_np_x\ e_{x+n} \quad \to \quad e_x = p_x + p_x\ e_{x+1}$

$\overset{o}{e}_{x:\overline{n|}} = \overset{o}{e}_{x:\overline{m|}} + {}_mp_x\ \overset{o}{e}_{x+m:\overline{n-m|}}$ for $m < n$

$e_{x:\overline{n|}} = e_{x:\overline{m|}} + {}_mp_x\ e_{x+m:\overline{n-m|}}$ for $m < n \quad \to \quad e_{x:\overline{n|}} = p_x + p_x\ e_{x+1:\overline{n-1|}}$

C6. MORTALITY LAWS

Gompertz's law: $\mu_x = Bc^x$ for $c > 1$ $\rightarrow$ ${}_tp_x = e^{-\frac{Bc^x(c^t-1)}{\log c}}$

Makeham's law: $\mu_x = A + Bc^x$ for $c > 1$ $\rightarrow$ ${}_tp_x = e^{-At-\frac{Bc^x(c^t-1)}{\log c}}$

Weibull distribution: $\mu_x = kx^n$ $\rightarrow$ ${}_tp_x = e^{-\frac{k\left((x+t)^{n+1}-x^{n+1}\right)}{n+1}}$

Exponential distribution: $\mu_x = \mu$ $\rightarrow$ ${}_tp_x = e^{-\mu t}$

(Constant force of mortality)

Uniform distribution: $\mu_x = \frac{1}{\omega-x}$ for $0 \le x \le \omega$ $\rightarrow$ ${}_tp_x = 1 - \frac{t}{\omega-x}$

Beta distribution: $\mu_x = \frac{\alpha}{\omega-x}$ for $0 \le x \le \omega$ $\rightarrow$ ${}_tp_x = \left(1 - \frac{t}{\omega-x}\right)^{\alpha}$

C7. APPROXIMATIONS

UDD between integral ages: $l_{x+s} = l_x - sd_x$ $\rightarrow$ ${}_sq_x = sq_x$ $\quad {}_sq_{x+t} = \frac{Sq_x}{1-tq_x}$ $\quad q_x = {}_sp_x\ \mu_{x+s}$

CFM between integral ages: $l_{x+s} = l_x \times (p_x)^s$ $\rightarrow$ ${}_sp_x = (p_x)^s$ $\quad {}_sp_{x+t} = (p_x)^s$ $\quad \mu_{x+s} = -\log p_x$

These are for $0 \le s, t \le 1$ and $0 \le s+t \le 1$.

C8. SELECT SURVIVAL MODEL

***k*-year select period:**

$q_{[x]+h} < q_{x+h}$ for $h < k$

$q_{[x]+h} = q_{x+h}$ for $h \ge k$

$p_{[x]+h} > p_{x+h}$ for $h < k$

$p_{[x]+h} = p_{x+h}$ for $h \ge k$

D. INSURANCE

D1. ACTUARIAL FUNCTIONS

Endowment: ${}_nE_x = A_{\substack{\ \ 1\\x:\overline{n|}}} = v^n{}_np_x$ $\qquad {}^2_nE_x = {}^2A_{\substack{\ \ 1\\x:\overline{n|}}} = (v^n)^2{}_np_x$

Insurance (Continuous): $\bar{A}_x = \int_0^\infty v^t{}_tp_x\ \mu_{x+t}dt$ $\qquad {}^2\bar{A}_x = \int_0^\infty (v^t)^2{}_tp_x\ \mu_{x+t}dt$

$\bar{A}^{\ 1}_{x:\overline{n|}} = \int_0^n v^t{}_tp_x\ \mu_{x+t}dt$ $\qquad {}^2\bar{A}^{\ 1}_{x:\overline{n|}} = \int_0^n (v^t)^2{}_tp_x\ \mu_{x+t}dt$

$\bar{A}_{x:\overline{n|}} = \int_0^n v^t{}_tp_x\ \mu_{x+t}dt + v^n{}_np_x$ $\qquad {}^2\bar{A}_{x:\overline{n|}} = \int_0^n (v^t)^2{}_tp_x\ \mu_{x+t}dt + (v^n)^2{}_np_x$

Insurance (Discrete): $A_x = \sum_{k=0}^{\infty} v^{k+1}{}_kp_x\ q_{x+k}$ $\qquad {}^2A_x = \sum_{k=0}^{\infty} (v^{k+1})^2{}_kp_x\ q_{x+k}$

$$A_{\overset{1}{x}:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x \; q_{x+k} \qquad {}^2A_{\overset{1}{x}:\overline{n}|} = \sum_{k=0}^{n-1} \left(v^{k+1}\right)^2 {}_kp_x \; q_{x+k}$$

$$A_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{k+1} {}_kp_x \; q_{x+k} + v^n {}_np_x \qquad {}^2A_{x:\overline{n}|} = \sum_{k=0}^{n-1} \left(v^{k+1}\right)^2 {}_kp_x \; q_{x+k} + (v^n)^2 {}_np_x$$

Insurance (mthly):

$$A^{(m)}_{\overset{1}{x}:\overline{n}|} = \sum_{k=0}^{nm-1} v^{k/m+1/m} {}_{k/m}p_x \; {}_{1/m}q_{x+k/m}$$

Relations:

$$\bar{A}_x = \bar{A}_{\overset{1}{x}:\overline{n}|} + {}_nE_x \; \bar{A}_{x+n} \qquad A_x = A_{\overset{1}{x}:\overline{n}|} + {}_nE_x \; A_{x+n}$$

$$\bar{A}_{x:\overline{n}|} = \bar{A}_{\overset{1}{x}:\overline{n}|} + {}_nE_x \qquad A_{x:\overline{n}|} = A_{\overset{1}{x}:\overline{n}|} + {}_nE_x$$

$${}_{n|}\bar{A}_x = {}_nE_x \; \bar{A}_{x+n} \qquad {}_{n|}A_x = {}_nE_x A_{x+n}$$

Recursive formulas:

$$A_x = vq_x + vp_x \; A_{x+1} \qquad {}^2A_x = v^2q_x + v^2p_x \; {}^2A_{x+1}$$

$$A_{\overset{1}{x}:\overline{n}|} = vq_x + vp_x \; A_{\overset{1}{x+1}:\overline{n-1}|} \qquad {}^2A_{\overset{1}{x}:\overline{n}|} = v^2q_x + v^2p_x \; {}^2A_{\overset{1}{x+1}:\overline{n-1}|}$$

D2. PRESENT VALUES

Policy	Death benefit paid at the moment of death	Death benefit paid at the end of the year of death
Whole life insurance	$Z = bv^{T_x}, T_x > 0$	$Z = bv^{K_x+1}, \quad K_x = 0, 1, 2, \ldots, \infty$
n-year term insurance	$Z = \begin{cases} bv^{T_x}, & T_x < n \\ 0, & T_x \geq n \end{cases}$	$Z = \begin{cases} bv^{K_x+1}, & K_x = 0, 1, 2, \ldots, n-1 \\ 0, & K_x = n, n+1, \ldots, \infty \end{cases}$
n-year endowment insurance	$Z = \begin{cases} bv^{T_x}, & T_x < n \\ bv^n, & T_x \geq n \end{cases}$	$Z = \begin{cases} bv^{K_x+1}, & K_x = 0, 1, 2, \ldots, n-1 \\ bv^n, & K_x = n, n+1, \ldots, \infty \end{cases}$
n-year pure endowment	$Z = \begin{cases} 0, & T_x < n \\ bv^n, & T_x \geq n \end{cases}$	
n-year deferred whole life insurance	$Z = \begin{cases} 0, & T_x < n \\ bv^{T_x}, & T_x \geq n \end{cases}$	$Z = \begin{cases} 0, & K_x = 0, 1, 2, \ldots, n-1 \\ bv^{K_x+1}, & K_x = n, n+1, \ldots, \infty \end{cases}$

D3. EXPECTED PRESENT VALUES

Policy	Death benefit paid at the moment of death	Death benefit paid at the end of the year of death
Whole life insurance	$E[Z] = b\bar{A}_x$	$E[Z] = bA_x$
n-year term insurance	$E[Z] = b\bar{A}_{\overset{1}{x}:\overline{n}|}$	$E[Z] = bA_{\overset{1}{x}:\overline{n}|}$

n-year endowment insurance	$E[Z] = b\bar{A}_{x:\overline{n}\vert}$	$E[Z] = bA_{x:\overline{n}\vert}$
n-year pure endowment	$E[Z] = b_nE_x$	
n-year deferred whole life insurance	$E[Z] = b_{n\vert}\bar{A}_x$	$E[Z] = b_{n\vert}A_x$

D4. VARIANCE OF PRESENT VALUES

Policy	Death benefit paid at the moment of death	Death benefit paid at the end of the year of death
Whole life insurance	$Var(Z) = b^2\left({}^2\bar{A}_x - (\bar{A}_x)^2\right)$	$Var(Z) = b^2\left({}^2A_x - (A_x)^2\right)$
n-year term insurance	$Var(Z) = b^2\left({}^2\bar{A}_{\overset{1}{x}:\overline{n}\vert} - \left(\bar{A}_{\overset{1}{x}:\overline{n}\vert}\right)^2\right)$	$Var(Z) = b^2\left({}^2A_{\overset{1}{x}:\overline{n}\vert} - \left(A_{\overset{1}{x}:\overline{n}\vert}\right)^2\right)$
n-year endowment insurance	$Var(Z) = b^2\left({}^2\bar{A}_{x:\overline{n}\vert} - (\bar{A}_{x:\overline{n}\vert})^2\right)$	$Var(Z) = b^2\left({}^2A_{x:\overline{n}\vert} - (A_{x:\overline{n}\vert})^2\right)$
n-year pure endowment	$Var(Z) = b^2\left({}^2_nE_x - ({}_nE_x)^2\right)$	
n-year deferred whole life insurance	$Var(Z) = b^2\left({}^2_{n\vert}\bar{A}_x - ({}_{n\vert}\bar{A}_x)^2\right)$	$Var(Z) = b^2\left({}^2_{n\vert}A_x - ({}_{n\vert}A_x)^2\right)$

D5. APPROXIMATIONS

UDD between integral ages:

$\bar{A}_x = \frac{i}{\delta}A_x$ $\quad$ $\bar{A}_{\overset{1}{x}:\overline{n}\vert} = \frac{i}{\delta}A_{\overset{1}{x}:\overline{n}\vert}$ $\quad$ $\bar{A}_{x:\overline{n}\vert} \neq \frac{i}{\delta}A_{x:\overline{n}\vert}$.

$A_x^{(m)} = \frac{i}{i^{(m)}}A_x$

${}^2\bar{A}_x = \frac{2i+i^2}{2\delta}\ {}^2A_x$

Claims acceleration approach:

$\bar{A}_x = (1+i)^{0.5}\ A_x$ $\quad$ $\bar{A}_{\overset{1}{x}:\overline{n}\vert} = (1+i)^{0.5}\ A_{\overset{1}{x}:\overline{n}\vert}$ $\quad$ $\bar{A}_{x:\overline{n}\vert} \neq (1+i)^{0.5}\ A_{x:\overline{n}\vert}$.

$A_x^{(m)} = (1+i)^{\frac{m-1}{2m}}\ A_x$

${}^2\bar{A}_x = (1+i)\ {}^2A_x$

E. ANNUITIES

E1. ACTUARIAL FUNCTIONS

Annuity (Continuous):

$\bar{a}_x = \int_0^\infty \left(\frac{1-v^t}{\delta}\right) {}_tp_x\ \mu_{x+t}dt$ $\quad$ ${}^2\bar{a}_x = \int_0^\infty \left(\frac{1-v^{2t}}{2\delta}\right) {}_tp_x\ \mu_{x+t}dt$

$\bar{a}_{x:\overline{n}\vert} = \int_0^n \left(\frac{1-v^t}{\delta}\right) {}_tp_x\ \mu_{x+t}dt + \left(\frac{1-v^n}{\delta}\right) {}_np_x$ $\quad$ ${}^2\bar{a}_{x:\overline{n}\vert} = \int_0^n \left(\frac{1-v^{2t}}{2\delta}\right) {}_tp_x\ \mu_{x+t}dt + \left(\frac{1-v^{2n}}{2\delta}\right) {}_np_x$

Annuity (Discrete):

$\ddot{a}_x = \sum_{k=0}^{\infty} \left(\frac{1-v^{k+1}}{d}\right) {}_kp_x\ q_{x+k}$ $\quad$ ${}^2\ddot{a}_x = \sum_{k=0}^{\infty} \left(\frac{1-v^{2k+2}}{2d-d^2}\right) {}_kp_x\ q_{x+k}$

$\ddot{a}_{x:\overline{n}\vert} = \sum_{k=0}^{n-1} \left(\frac{1-v^{k+1}}{d}\right) {}_kp_x\ q_{x+k} + \left(\frac{1-v^n}{d}\right) {}_np_x$ $\quad$ ${}^2\ddot{a}_{x:\overline{n}\vert} = \sum_{k=0}^{n-1} \left(\frac{1-v^{2k+2}}{2d-d^2}\right) {}_kp_x\ q_{x+k} + \left(\frac{1-v^{2n}}{2d-d^2}\right) {}_np_x$

Simpler formulas:

$\bar{a}_x = \int_0^\infty v^t \, {}_tp_x dt$ $\quad$ ${}^2\bar{a}_x = \int_0^\infty v^{2t} \, {}_tp_x dt$

$\bar{a}_{x:\overline{n}|} = \int_0^n v^t \, {}_tp_x dt$ $\quad$ ${}^2\bar{a}_{x:\overline{n}|} = \int_0^n v^{2t} \, {}_tp_x dt$

$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \, {}_kp_x$ $\quad$ ${}^2\ddot{a}_x = \sum_{k=0}^{\infty} v^{2k} \, {}_kp_x$

$\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^k \, {}_kp_x$ $\quad$ ${}^2\ddot{a}_{x:\overline{n}|} = \sum_{k=0}^{n-1} v^{2k} \, {}_kp_x$

$\ddot{a}^{(m)}_{x:\overline{n}|} = \sum_{k=0}^{nm-1} \frac{1}{m} v^{k/m} \, {}_{k/m}p_x$

Relations:

$\bar{a}_x = \bar{a}_{x:\overline{n}|} + {}_nE_x \; \bar{a}_{x+n}$ $\quad$ $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + {}_nE_x \; \ddot{a}_{x+n}$

${}_{n|}\bar{a}_x = {}_nE_x \; \bar{a}_{x+n}$ $\quad$ ${}_{n|}\ddot{a}_x = {}_nE_x \; \ddot{a}_{x+n}$

$\bar{a}_{\overline{x:\overline{n}|}} = \bar{a}_{\overline{n}|} + {}_nE_x \; \bar{a}_{x+n}$ $\quad$ $\ddot{a}_{\overline{x:\overline{n}|}} = \ddot{a}_{\overline{n}|} + {}_nE_x \; \ddot{a}_{x+n}$

$\ddot{a}_x = 1 + a_x$

$\ddot{a}_{x:\overline{n}|} = 1 + a_{x:\overline{n-1}|} = 1 + a_{x:\overline{n}|} - {}_nE_x$

Recursive formulas:

$\ddot{a}_x = 1 + vp_x \; \ddot{a}_{x+1}$ $\quad$ ${}^2\ddot{a}_x = 1 + v^2p_x \; {}^2\ddot{a}_{x+1}$

$\ddot{a}_{x:\overline{n}|} = 1 + vp_x \; \ddot{a}_{x+1:\overline{n-1}|}$ $\quad$ ${}^2\ddot{a}_{x:\overline{n}|} = 1 + v^2p_x \; {}^2\ddot{a}_{x+1:\overline{n-1}|}$

Insurance to annuity:

$\bar{a}_x = \frac{1-\bar{A}_x}{\delta}$ $\quad$ ${}^2\bar{a}_x = \frac{1-{}^2\bar{A}_x}{2\delta}$

$\bar{a}_{x:\overline{n}|} = \frac{1-\bar{A}_{x:\overline{n}|}}{\delta}$ $\quad$ ${}^2\bar{a}_{x:\overline{n}|} = \frac{1-{}^2\bar{A}_{x:\overline{n}|}}{2\delta}$

$\ddot{a}_x = \frac{1-A_x}{d}$ $\quad$ ${}^2\ddot{a}_x = \frac{1-{}^2A_x}{2d-d^2}$

$\ddot{a}_{x:\overline{n}|} = \frac{1-A_{x:\overline{n}|}}{d}$ $\quad$ ${}^2\ddot{a}_{x:\overline{n}|} = \frac{1-{}^2A_{x:\overline{n}|}}{2d-d^2}$

$\ddot{a}^{(m)}_{x:\overline{n}|} = \frac{1-A^{(m)}_{x:\overline{n}|}}{d^{(m)}}$

E2. PRESENT VALUES

Policy	Payments made continuously	Payments at the beginning of each year
Whole life annuity	$Y = b\left(\frac{1-v^{T_x}}{\delta}\right), T_x > 0$	$Y = b\left(\frac{1-v^{K_x+1}}{d}\right), \quad K_x = 0,1,2,\ldots,\infty$
n-year term annuity	$Y = \begin{cases} b\left(\frac{1-v^{T_x}}{\delta}\right), & T_x < n \\ b\left(\frac{1-v^n}{\delta}\right), & T_x \geq n \end{cases}$	$Y = \begin{cases} b\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{1-v^n}{d}\right), & K_x = n, n+1,\ldots,\infty \end{cases}$
n-year certain whole life annuity	$Y = \begin{cases} b\left(\frac{1-v^n}{\delta}\right), & T_x < n \\ b\left(\frac{1-v^{T_x}}{\delta}\right), & T_x \geq n \end{cases}$	$Y = \begin{cases} b\left(\frac{1-v^n}{d}\right), & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = n, n+1,\ldots,\infty \end{cases}$

n-year deferred whole life annuity	$Y = \begin{cases} 0, & T_x < n \\ b\left(\frac{v^n - v^{T_x}}{\delta}\right), & T_x \geq n \end{cases}$	$Y = \begin{cases} 0, & K_x = 0,1,2,\ldots,n-1 \\ b\left(\frac{v^n - v^{K_x+1}}{d}\right), & K_x = n, n+1, \ldots, \infty \end{cases}$

E3. EXPECTED PRESENT VALUES

Policy	Payments made continuously	Payments at the beginning of each year
Whole life annuity	$E[Y] = b\bar{a}_x$	$E[Y] = b\ddot{a}_x$
n-year term annuity	$E[Y] = b\bar{a}_{x:\overline{n}\rvert}$	$E[Y] = b\ddot{a}_{x:\overline{n}\rvert}$
n-year certain whole life annuity	$E[Y] = b\bar{a}_{\overline{x:\overline{n}\rvert}}$	$E[Y] = b\ddot{a}_{\overline{x:\overline{n}\rvert}}$
n-year deferred whole life annuity	$E[Y] = b_{n\rvert}\bar{a}_x$	$E[Y] = b_{n\rvert}\ddot{a}_x$

E4. VARIANCE OF PRESENT VALUES

Policy	Payments made continuously	Payments at the beginning of each year
Whole life annuity	$Var(Y) = b^2 \frac{{}^2\bar{A}_x - (\bar{A}_x)^2}{\delta^2}$	$Var(Y) = b^2 \frac{{}^2A_x - (A_x)^2}{d^2}$
n-year term annuity	$Var(Y) = b^2 \frac{{}^2\bar{A}_{x:\overline{n}\rvert} - (\bar{A}_{x:\overline{n}\rvert})^2}{\delta^2}$	$Var(Y) = b^2 \frac{{}^2A_{x:\overline{n}\rvert} - (A_{x:\overline{n}\rvert})^2}{d^2}$

E5. APPROXIMATIONS

UDD between integral ages:

$$\bar{a}_x = \frac{id}{\delta^2}\ddot{a}_x - \frac{i-\delta}{\delta^2}$$

$$\ddot{a}_x^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_x - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}$$

$$\ddot{a}_{x:\overline{n}\rvert}^{(m)} = \frac{id}{i^{(m)}d^{(m)}}\ddot{a}_{x:\overline{n}\rvert} - \frac{i-i^{(m)}}{i^{(m)}d^{(m)}}\left(1 - {}_nE_x\right)$$

Woolhouse formula, 2 terms:

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2}$$

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m}$$

$$\ddot{a}_{x:\overline{n}\rvert}^{(m)} \approx \ddot{a}_{x:\overline{n}\rvert} - \frac{m-1}{2m}\left(1 - {}_nE_x\right)$$

Woolhouse formula, 3 terms:

$$\bar{a}_x \approx \ddot{a}_x - \frac{1}{2} - \frac{1}{12}\left(\mu_x + \delta\right)$$

$$\ddot{a}_x^{(m)} \approx \ddot{a}_x - \frac{m-1}{2m} - \frac{m^2-1}{12m^2}\left(\mu_x + \delta\right)$$ where $\mu_x \approx -\frac{1}{2}\left(\log p_{x-1} + \log p_x\right)$ if not given.

$$\ddot{a}_{x:\overline{n}\rvert}^{(m)} \approx \ddot{a}_{x:\overline{n}\rvert} - \frac{m-1}{2m}\left(1 - {}_nE_x\right) - \frac{m^2-1}{12m^2}\left(\mu_x + \delta - {}_nE_x\left(\mu_{x+n} + \delta\right)\right)$$

F. PREMIUMS

F1. ACTUARIAL FUNCTIONS

Premium (Discrete): $P_x = \frac{A_x}{\ddot{a}_x}$ $\quad P^{1}_{x:\overline{n}|} = \frac{A^{1}_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$ $\quad P_{x:\overline{n}|} = \frac{A_{x:\overline{n}|}}{\ddot{a}_{x:\overline{n}|}}$

Premium (Continuous): $\bar{P}\left(\bar{A}_x\right) = \frac{\bar{A}_x}{\bar{a}_x}$ $\quad \bar{P}\left(\bar{A}^{1}_{x:\overline{n}|}\right) = \frac{\bar{A}^{1}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$ $\quad \bar{P}\left(\bar{A}_{x:\overline{n}|}\right) = \frac{\bar{A}_{x:\overline{n}|}}{\bar{a}_{x:\overline{n}|}}$

F2. EQUIVALENCE PRINCIPLE

Loss-at-issue:

$${}_0L^n = PV(\text{Benefits}) - PV(\text{Premiums})$$

$${}_0L^g = PV(\text{Benefits}) + PV(\text{Expenses}) - PV(\text{Premiums})$$

Equivalence principle:

$$\mathrm{E}[{}_0L] = EPV(\text{Benefits}) - EPV(\text{Premiums}) = 0$$

$$\mathrm{E}[{}_0L^g] = EPV(\text{Benefits}) + EPV(\text{Expenses}) - EPV(\text{Premiums}) = 0$$

F3. EXPECTATION AND VARIANCE

Fully continuous whole life insurance	Fully discrete whole life insurance
${}_0L^g = (b+E)\,v^{T_x} - (G-e)\left(\frac{1-v^{T_x}}{\delta}\right), \quad T_x \geq 0$	${}_0L^g = (b+E)\,v^{K_x+1} - (G-e)\left(\frac{1-v^{K_x+1}}{d}\right), \quad K_x = 0,1,2,\ldots,\infty$
${}_0L^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$	${}_0L^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$
$E\left[{}_0L^g\right] = (b+E)\,\bar{A}_x - (G-e)\,\bar{a}_x$	$E\left[{}_0L^g\right] = (b+E)\,A_x - (G-e)\,\ddot{a}_x$
$Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_x - \left(\bar{A}_x\right)^2\right)$	$Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_x - \left(A_x\right)^2\right)$
Fully continuous n-year endowment insurance	**Fully discrete n-year endowment insurance**
${}_0L^g = \begin{cases}(b+E)\,v^{T_x} - (G-e)\left(\frac{1-v^{T_x}}{\delta}\right), & T_x < n \\ (b+E)\,v^n - (G-e)\left(\frac{1-v^n}{\delta}\right), & T_x \geq n\end{cases}$	${}_0L^g = \begin{cases}(b+E)\,v^{K_x+1} - (G-e)\left(\frac{1-v^{K_x+1}}{d}\right), & K_x = 0,1,\ldots,n-1 \\ (b+E)\,v^n - (G-e)\left(\frac{1-v^n}{d}\right), & K_x = n,n+1,\ldots,\infty\end{cases}$
${}_0L^g = \left(b+E+\frac{G-e}{\delta}\right)Z - \frac{G-e}{\delta}$	${}_0L^g = \left(b+E+\frac{G-e}{d}\right)Z - \frac{G-e}{d}$
$E\left[{}_0L^g\right] = (b+E)\,\bar{A}_{x:\overline{n}|} - (G-e)\,\bar{a}_{x:\overline{n}|}$	$E\left[{}_0L^g\right] = (b+E)\,A_{x:\overline{n}|} - (G-e)\,\ddot{a}_{x:\overline{n}|}$
$Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x:\overline{n}|} - \left(\bar{A}_{x:\overline{n}|}\right)^2\right)$	$Var\left({}_0L^g\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x:\overline{n}|} - \left(A_{x:\overline{n}|}\right)^2\right)$

F4. PORTFOLIO PERCENTILE PRINCIPLE

Aggregate losses: $S = L_1^g + L_2^g + \cdots + L_n^g \dot{\sim} N\left(\mu, \sigma^2\right)$

Mean: $\mu = E[S] = nE\left[{}_0L^g\right]$

Variance: $\sigma^2 = \text{Var}(S) = n\,\text{Var}\left({}_0L^g\right)$

Find premium such that: $\Pr(S<0) \approx N\left(\frac{0-nE[{}_0L^g]}{\sqrt{n\,\text{Var}({}_0L^g)}}\right) = \alpha \quad \rightarrow \quad \frac{0-nE[{}_0L^g]}{\sqrt{n\,\text{Var}({}_0L^g)}} = z_\alpha$

G. RESERVES

G1. PROSPECTIVE FORMULA

Net premium reserve: ${}_tV^n = E\left[{}_tL \mid T_x \geq t\right] = EPV_t\ (\text{Benefits}) - EPV_t\ (\text{Net Premiums})$

Gross premium reserve: ${}_tV^g = E\left[{}_tL^g \mid T_x \geq t\right] = EPV_t\ (\text{Benefits}) + EPV_t\ (\text{Expenses}) - EPV_t\ (\text{Gross Premiums})$

Expense reserve: ${}_tV^e = {}_tV^g - {}_tV^n = EPV_t(\text{Expenses}) - EPV_t(\text{Expense Loadings})$

G2. EXPECTATION AND VARIANCE

Fully continuous whole life insurance	Fully discrete whole life insurance
${}_tL^g = (b+E)\,v^{T_{x+t}} - (G-e)\left(\frac{1-v^{T_{x+t}}}{\delta}\right), \quad T_{x+t} \geq 0$	${}_kL^g = (b+E)\,v^{K_{x+k}+1} - (G-e)\left(\frac{1-v^{K_{x+k}+1}}{d}\right),$ $K_{x+k} = 0, 1, 2, \ldots, \infty$
${}_tL^g = \left(b+E+\frac{G-e}{\delta}\right) Z - \frac{G-e}{\delta}$	${}_kL^g = \left(b+E+\frac{G-e}{d}\right) Z - \frac{G-e}{d}$
$E\left[{}_tL^g \mid T_x \geq t\right] = (b+E)\bar{A}_{x+t} - (G-e)\bar{a}_{x+t}$	$E\left[{}_kL^g \mid K_x \geq k\right] = (b+E)A_{x+k} - (G-e)\ddot{a}_{x+k}$
$Var\left({}_tL^g \mid T_x \geq t\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x+t} - \left(\bar{A}_{x+t}\right)^2\right)$	$Var\left({}_kL^g \mid K_x \geq k\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x+k} - \left(A_{x+k}\right)^2\right)$

Fully continuous n-year endowment insurance	Fully discrete n-year endowment insurance
${}_tL^g = \begin{cases} (b+E)\,v^{T_{x+t}} - (G-e)\left(\frac{1-v^{T_{x+t}}}{\delta}\right), & T_{x+t} < n-t \\ (b+E)\,v^{n-t} - (G-e)\left(\frac{1-v^{n-t}}{\delta}\right), & T_{x+t} \geq n-t \end{cases}$	${}_kL^g = \begin{cases} (b+E)\,v^{K_{x+k}+1} - (G-e)\left(\frac{1-v^{K_{x+k}+1}}{d}\right), & K_{x+k} = 0, \ldots, n-k-1 \\ (b+E)\,v^{n-k} - (G-e)\left(\frac{1-v^{n-k}}{d}\right), & K_{x+k} = n-k, n-k+1, \ldots \end{cases}$
${}_tL^g = \left(b+E+\frac{G-e}{\delta}\right) Z - \frac{G-e}{\delta}$	${}_kL^g = \left(b+E+\frac{G-e}{d}\right) Z - \frac{G-e}{d}$
$E\left[{}_tL^g \mid T_x \geq t\right] = (b+E)\bar{A}_{x+t:\overline{n-t}\rvert} - (G-e)\bar{a}_{x+t:\overline{n-t}\rvert}$	$E\left[{}_kL^g \mid K_x \geq k\right] = (b+E)A_{x+k:\overline{n-k}\rvert} - (G-e)\ddot{a}_{x+k:\overline{n-k}\rvert}$
$Var\left({}_tL^g \mid T_x \geq t\right) = \left(b+E+\frac{G-e}{\delta}\right)^2\left({}^2\bar{A}_{x+t:\overline{n-t}\rvert} - \left(\bar{A}_{x+t:\overline{n-t}\rvert}\right)^2\right)$	$Var\left({}_kL^g \mid K_x \geq k\right) = \left(b+E+\frac{G-e}{d}\right)^2\left({}^2A_{x+k:\overline{n-k}\rvert} - \left(A_{x+k:\overline{n-k}\rvert}\right)^2\right)$

G3. RECURSIVE FORMULA

Net premium reserve: $\left({}_kV + P\right)(1+i) = bq_{x+k} + {}_{k+1}V\ p_{x+k}$

Gross premium Reserve: $\left({}_kV^g + G - e\right)(1+i) = (b+E)\,q_{x+k} + {}_{k+1}V^g\ p_{x+k}$

For reserves between premium dates: $\left({}_kV + P\right)(1+i_k)^s = b\ v^{1-s}\ {}_sq_{x+k} + {}_{k+s}V\ {}_sp_{x+k}$ where $0 < s < 1$.

${}_{k+s}V\,(1+i_k)^{1-s} = b\ {}_{1-s}q_{x+k+s} + {}_{k+1}V\ {}_{1-s}p_{x+k+s}$ where $0 < s < 1$.

For a special policy that pays $FA + {}_kV$ upon death: $\left({}_{k-1}V + P\right)(1+i) = \left(FA + {}_kV\right)\ q_{x+k-1} + {}_kV\ p_{x+k-1}$

Rearrange: ${}_kV = \left({}_{k-1}V + P\right)(1+i) - FA\ q_{x+k-1}$

Obtain the formula: ${}_kV = P\ddot{a}_{\overline{k}|}(1+i)^k - FA\sum_{j=1}^{k} q_{x+j-1}(1+i)^{k-j}$

G4. FPT RESERVES

First year premium: $E\left[{}_0L^{FPT}\right] = vq_x - \alpha = 0 \quad\rightarrow\quad \alpha = vq_x$

Renewal year premium:

${}_1V^{FPT} = A_{x+1} - \beta\ddot{a}_{x+1} = 0$	$\rightarrow\ \beta = \frac{A_{x+1}}{\ddot{a}_{x+1}}$	whole life
${}_1V^{FPT} = A^{\ 1}_{x+1:\overline{n-1}|} - \beta\ddot{a}_{x+1:\overline{n-1}|} = 0$	$\rightarrow\ \beta = \frac{A^{\ 1}_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}}$	n-year term
${}_1V^{FPT} = A_{x+1:\overline{n-1}|} - \beta\ddot{a}_{x+1:\overline{n-1}|} = 0$	$\rightarrow\ \beta = \frac{A_{x+1:\overline{n-1}|}}{\ddot{a}_{x+1:\overline{n-1}|}}$	n-year endowment