Exam FM
Formula & Review Sheet
(updated 06/29/2023)

TIME VALUE OF MONEY

Accumulation and Amount Functions: $A(t) = Ka(t)$, $A(0) = K$, $a(0) = 1$

Effective Interest Rate: $i_t = \dfrac{a(t) - a(t-1)}{a(t-1)} = \dfrac{A(t) - A(t-1)}{A(t-1)}$

Simple Interest: $a(t) = 1 + it$ $\quad i_t = \dfrac{i}{1 + i(t-1)}$

Compound Interest: $a(t) = (1+i)^t$ $\quad i_t = i$

Effective Discount Rate: $d_t = \dfrac{a(t) - a(t-1)}{a(t)} = \dfrac{A(t) - A(t-1)}{A(t)}$

Discount Rate: $d = \dfrac{i}{1+i} = 1 - v = iv$ $\quad v = (1+i)^{-1}$ $\quad \dfrac{1}{d} - \dfrac{1}{i} = 1$

Nominal Rates: $\left(1 + \dfrac{i^{(m)}}{m}\right)^m = 1 + i$ $\quad \left(1 - \dfrac{d^{(m)}}{m}\right)^m = 1 - d$

$$i^{(m)} = m\left[(1+i)^{\frac{1}{m}} - 1\right] \quad d^{(m)} = m\left[1 - (1-d)^{\frac{1}{m}}\right]$$

Force of Interest: $\delta_t = \dfrac{a'(t)}{a(t)} = \dfrac{A'(t)}{A(t)} = \dfrac{d}{dt}\ln a(t)$ $\quad a(t) = e^{\int_0^t \delta_r dr}$

Constant Force of Interest: $e^\delta = 1 + i$ $\quad \delta = \ln(1+i)$

PV of \$1 Due in ***t*** Years:

$$PV = \frac{1}{a(t)} = (1+i)^{-t} = v^t = e^{-\delta t} = (1-d)^t$$
$$= \left(1 + \frac{i^{(m)}}{m}\right)^{-mt} = \left(1 - \frac{d^{(m)}}{m}\right)^{mt}$$

AV at time ***t*** of \$1 invested at time 0:

$$AV = a(t) = (1+i)^t = e^{\delta t} = (1-d)^{-t} = \left(1 + \frac{i^{(m)}}{m}\right)^{mt} = \left(1 - \frac{d^{(m)}}{m}\right)^{-mt}$$

AV at time t_2 for \$1 Invested at time t_1: $AV = \dfrac{a(t_2)}{a(t_1)} = e^{\int_{t_1}^{t_2} \delta_t dt}$

ANNUITIES

Annuity-Immediate: (PV one period before first payment, AV at time of last payment)

$$PV = a_{\overline{n}|} = v + v^2 + \cdots + v^n = \frac{1 - v^n}{i}$$

$$AV = s_{\overline{n}|} = 1 + (1+i) + \cdots + (1+i)^{n-1} = \frac{(1+i)^n - 1}{i} = a_{\overline{n}|}(1+i)^n$$

Annuity-Due: (PV at time of first payment, AV one period after last payment)

$$PV = \ddot{a}_{\overline{n}|} = 1 + v + \cdots + v^{n-1} = \frac{1 - v^n}{d} = (1+i)a_{\overline{n}|} = \left(\frac{i}{d}\right)a_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

$$AV = \ddot{s}_{\overline{n}|} = (1+i) + (1+i)^2 + \cdots + (1+i)^n = \frac{(1+i)^n - 1}{d}$$
$$= (1+i)s_{\overline{n}|} = \left(\frac{i}{d}\right)s_{\overline{n}|} = s_{\overline{n+1}|} - 1$$

Continuous Annuity:

$$PV = \bar{a}_{\overline{n}|} = \frac{1-v^n}{\delta} = \left(\frac{i}{\delta}\right) a_{\overline{n}|} = \int_0^n v^t dt = \int_0^n e^{-\delta t} dt$$

$$AV = \bar{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{\delta} = \left(\frac{i}{\delta}\right) s_{\overline{n}|} = \int_0^n (1+i)^{n-t} dt = \int_0^n e^{\delta(n-t)} dt$$

Deferred Annuity:

$$_{m}|a_{\overline{n}|} = {}_{m+1}|\ddot{a}_{\overline{n}|} = v^m a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}$$

Perpetuity:

$$a_{\overline{\infty}|} = \frac{1}{i} \qquad \ddot{a}_{\overline{\infty}|} = \frac{1}{d} \qquad \bar{a}_{\overline{\infty}|} = \frac{1}{\delta}$$

Increasing Annuity - Payments in Arithmetic Progression:

$$(Ia)_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i} \qquad (Is)_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{i} = \frac{s_{\overline{n+1}|} - (n+1)}{i}$$

$$(I\ddot{a})_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{d} \qquad (I\ddot{s})_{\overline{n}|} = \frac{\ddot{s}_{\overline{n}|} - n}{d} = \frac{s_{\overline{n+1}|} - (n+1)}{d}$$

Decreasing Annuity - Payments in Arithmetic Progression:

$$(Da)_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{i} \qquad (Ds)_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{i}$$

$$(D\ddot{a})_{\overline{n}|} = \frac{n - a_{\overline{n}|}}{d} \qquad (D\ddot{s})_{\overline{n}|} = \frac{n(1+i)^n - s_{\overline{n}|}}{d}$$

Increasing Perpetuity - Payments in Arithmetic Progression:

$$(Ia)_{\overline{\infty}|} = \frac{1}{i} + \frac{1}{i^2} = \frac{1}{id} \qquad (I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2}$$

***m*-thly Annuity**:

$$a^{(m)}_{\overline{n}|} = \frac{1-v^n}{i^{(m)}} \qquad \ddot{a}^{(m)}_{\overline{n}|} = \frac{1-v^n}{d^{(m)}}$$

$$(Ia)^{(m)}_{\overline{n}|} = \frac{\ddot{a}_{\overline{n}|} - nv^n}{i^{(m)}} \qquad (I^{(m)}a)^{(m)}_{\overline{n}|} = \frac{\ddot{a}^{(m)}_{\overline{n}|} - nv^n}{i^{(m)}}$$

Continuously Increasing Annuity:

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \int_0^n tv^t dt = \int_0^n te^{-\delta t} dt$$

$$(\bar{I}\bar{s})_{\overline{n}|} = \frac{\bar{s}_{\overline{n}|} - n}{\delta} = \int_0^n t(1+i)^{n-t} dt = \int_0^n te^{\delta(n-t)} dt$$

Annuity with Varying Payments and Varying Force of Interest:

$PV = \int_0^n f(t)e^{-\delta t} dt$ If force of interest varies: $PV = \int_0^n f(t)e^{-\int_0^t \delta_r dr} dt$

$AV = \int_0^n f(t)e^{\delta(n-t)} dt$ If force of interest varies: $FV = \int_0^n f(t)e^{\int_t^n \delta_r dr} dt$

Geometric Annuity-Immediate: 1st Payment is 1. Subsequent payments increase by a factor of $(1+k)$.

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{i-k} \qquad FV = \frac{(1+i)^n - (1+k)^n}{i-k}$$

Geometric Annuity-Due: 1st Payment is 1. Subsequent payments increase by a factor of $(1+k)$.

$$PV = \frac{1 - \left(\frac{1+k}{1+i}\right)^n}{d-kv} \qquad FV = \frac{(1+i)^n - (1+k)^n}{d-kv}$$

Sum of a Geometric Series:

$$a + ar + ar^2 + \cdots + ar^{n-1} = a\left(\frac{1-r^n}{1-r}\right)$$

LOANS

Amortization Method: Level payment P, Outstanding Balance B_t, Principal Repayment $Prin_t$

$L = B_0 = Pa_{\overline{n}|}$ $\quad P = Prin_t + I_t$ $\quad I_t = iB_{t-1}$

$Prin_t = P - I_t = Pv^{n-t+1}$ $\quad Prin_{t+s} = Prin_t(1+i)^s$ $\quad I_t = P - Prin_t = P(1 - v^{n-t+1})$

$\sum_{t=1}^{n} I_t = nP - L$ $\quad \sum_{t=1}^{n} Prin_t = L$ $\quad B_t = B_{t-1} - Prin_t$

$B_t = Pa_{\overline{n-t}|}$ (Prospective) $\quad B_t = L(1+i)^t - Ps_{\overline{t}|}$ (Retrospective)

BONDS

Price Formulas: Number of coupon payments n, Coupon rate r, Face amount F, Maturity value C

$P = Fra_{\overline{n}|} + Cv^n$ $\quad P = C + (Fr - Ci)a_{\overline{n}|}$

$P = K + \frac{g}{i}(C - K)$, where $K = Cv^n$ and $g = \frac{Fr}{C}$

Callable Bonds: To calculate appropriate price:

If Bond is sold at a **Premium**, assume Earliest Redemption date

If Bond is sold at a **Discount**, assume Latest Redemption date

Exception: If a premium bond has a call premium, calculate price both ways (using Earliest Redemption date and using Latest Redemption date).

Use the smaller of the 2 calculated values.

If $g > i$, then $P > C$, and **Premium** $= P - C = (Fr - Ci)a_{\overline{n}|} = (Cg - Ci)a_{\overline{n}|}$

If $g < i$, then $P < C$, and **Discount** $= C - P = (Ci - Fr)a_{\overline{n}|} = (Ci - Cg)a_{\overline{n}|}$

Book Value at time t: $B_t = Fra_{\overline{n-t}|} + Cv^{n-t}$

Interest Earned: $I_t = iB_{t-1}$

If $g > i$, **Premium Amortized** at time $t = Fr - I_t = B_{t-1} - B_t = (Fr - Ci)v^{n-t+1}$

If $g < i$, **Discount Accumulated/Accrued** at time $t = I_t - Fr = B_t - B_{t-1} = (Ci - Fr)v^{n-t+1}$

GENERAL CASH FLOW

Reinvestment Rates: Interest rate i, Reinvestment rate i'

A single deposit of \$1, $AV = 1 + is_{\overline{n}|i'}$

Deposits of \$1 at beginning of each year, $AV = n + i(Is)_{\overline{n}|i'}$

Spot Rates:

Effective annual rate on investment for t years: r_t

Price of a t-year zero-coupon Bond: $P_t = (1+r_t)^{-t}$

Forward Rates: (Annual Effective)

Forward rate for the period $(t, t+1)$: $f_{[t,t+1]} = \dfrac{(1+r_{t+1})^{t+1}}{(1+r_t)^t} - 1 = \dfrac{P_t}{P_{t+1}} - 1$

Forward rate for the period $(t, t+m)$: $(1+r_t)^t(1+f_{[t,t+m]})^m = (1+r_{t+m})^{t+m}$

$$f_{[t,t+m]} = \left(\frac{(1+r_{t+1})^{t+m}}{(1+r_t)^t}\right)^{\frac{1}{m}} - 1$$

Inflation Rate: Real rate of interest i', Inflation rate r

$1+i = (1+i')(1+r)$ $\quad$ $i = i' + r + i'r$

Duration:

$$D_{\text{mac}}(i) = \frac{\sum tA_tv^t}{\sum A_tv^t} = -\frac{\frac{d}{d\delta}P}{P} \qquad D_{\text{mod}}(i) = \frac{\sum tA_tv^{t+1}}{\sum A_tv^t} = -\frac{\frac{d}{di}P}{P} = \frac{D_{\text{mac}}}{1+i}$$

Perpetuity: $D_{\text{mac}} = \dfrac{1+i}{i} = \dfrac{1}{d}$ $\quad$ Mortgage or Level Annuity: $D_{\text{mac}} = \dfrac{(Ia)_{\overline{n}|}}{a_{\overline{n}|}}$

Bond: $D_{\text{mac}} = \dfrac{Fr(Ia)_{\overline{n}|} + nCv^n}{P}$ $\quad$ Bond Sold at Par: $D_{\text{mac}} = \ddot{a}_{\overline{n}|}$

Convexity:

$$C_{\text{mac}} = \frac{\sum t^2A_tv^t}{\sum A_tv^t} = \frac{\frac{d^2}{d\delta^2}P}{P} \qquad C_{\text{mod}} = \frac{\sum t(t+1)A_tv^{t+2}}{\sum A_tv^t} = \frac{\frac{d^2}{di^2}P}{P} = \frac{C_{\text{mac}} + D_{\text{mac}}}{(1+i)^2}$$

Duration of a Portfolio: D_t and P_t are duration and price of t^{th} components of Portfolio

$$\text{D(Portfolio)} = \frac{D_1P_1 + D_2P_2 + \cdots + D_nP_n}{P_1 + P_2 + \cdots + P_n}$$

First-Order Modified Price Approximation: $P(i) \approx P(i_0) - D_{\text{mod}}(i_0)P(i_0)(i-i_0)$

First-Order Macaulay Price Approximation: $P(i) \approx P(i_0)\left(\dfrac{1+i_0}{1+i}\right)^{D_{\text{mac}}(i_0)}$

Second-Order Modified Price Approximation: $P(i) \approx P(i_0) - D_{\text{mod}}(i_0)P(i_0)(i-i_0) + C_{\text{mod}}(i_0)P(i_0)\dfrac{(i-i_0)^2}{2}$

IMMUNIZATION

Requirements for Redington Immunization: If same interest rate applies to all CF's

(i) PV(Assets) = PV(Liabilities)	(i) $P_A = P_L$	(i) $\sum A_tv^t = \sum L_tv^t$
(ii) D_{mod}(Assets) = D_{mod}(Liabilities)	(ii) $P'_A = P'_L$	(ii) $\sum tA_tv^t = \sum tL_tv^t$
(iii) C_{mod}(Assets) > C_{mod}(Liabilities)	(iii) $P''_A > P''_L$	(iii) $\sum t^2A_tv^t > \sum t^2L_tv^t$

Full Immunization: (i) and (ii) as above, (iii) There is a single liability CF that is matched (in PV and D_{mod}) with 2 or more asset CFs, including at least one that occurs before and at least one that occurs after the liability CF.

Exact Matching or Dedication: Match both the amount and the time of Asset Cash Flows and Liability Cash Flows.