ACTEX Learning

Study Manual for Exam ALTAM

2nd Edition

Johnny Li, PhD, FSA
Andrew Ng, PhD, FSA

An SOA Exam

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ACTEX Learning

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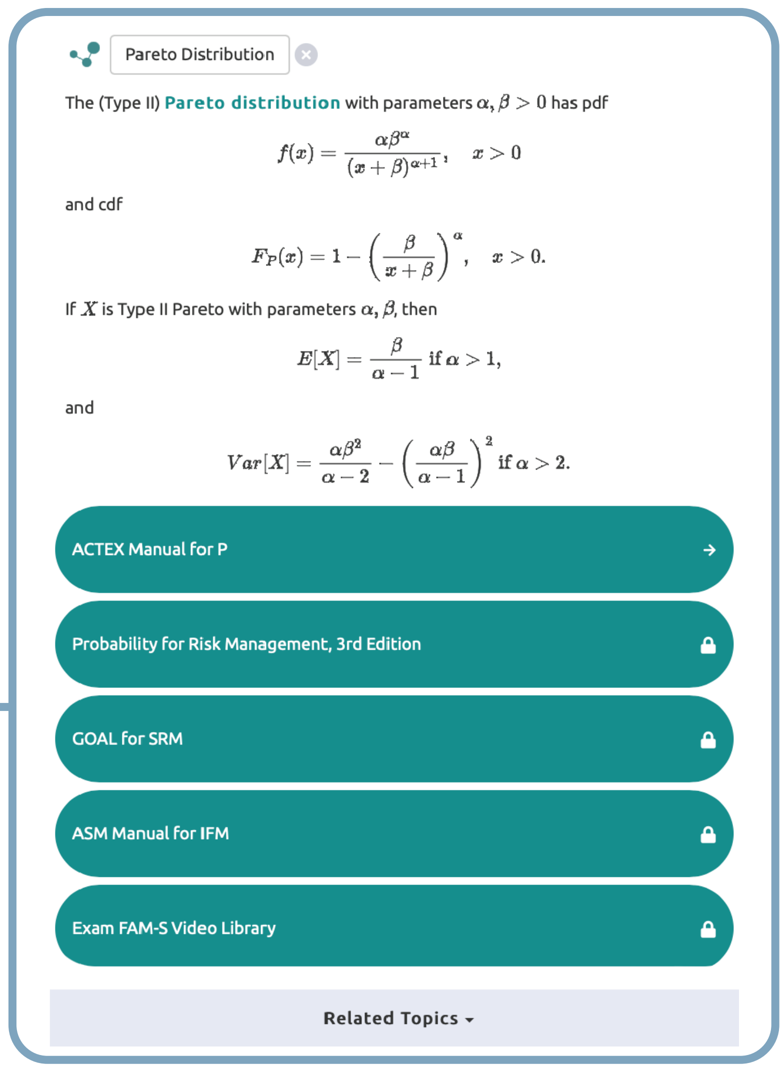

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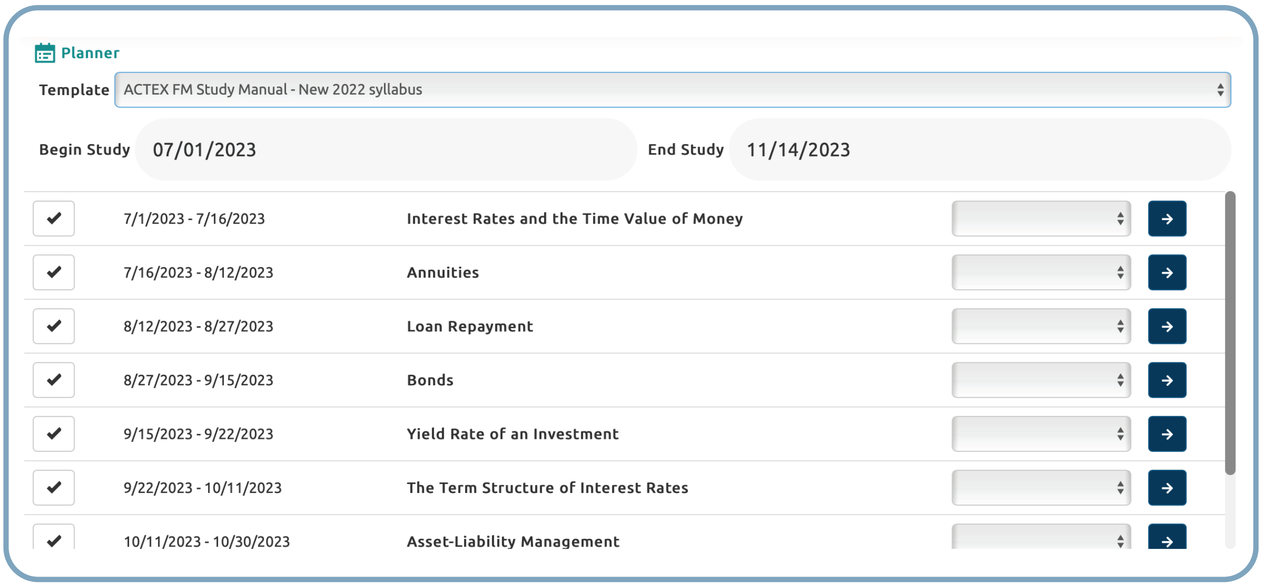

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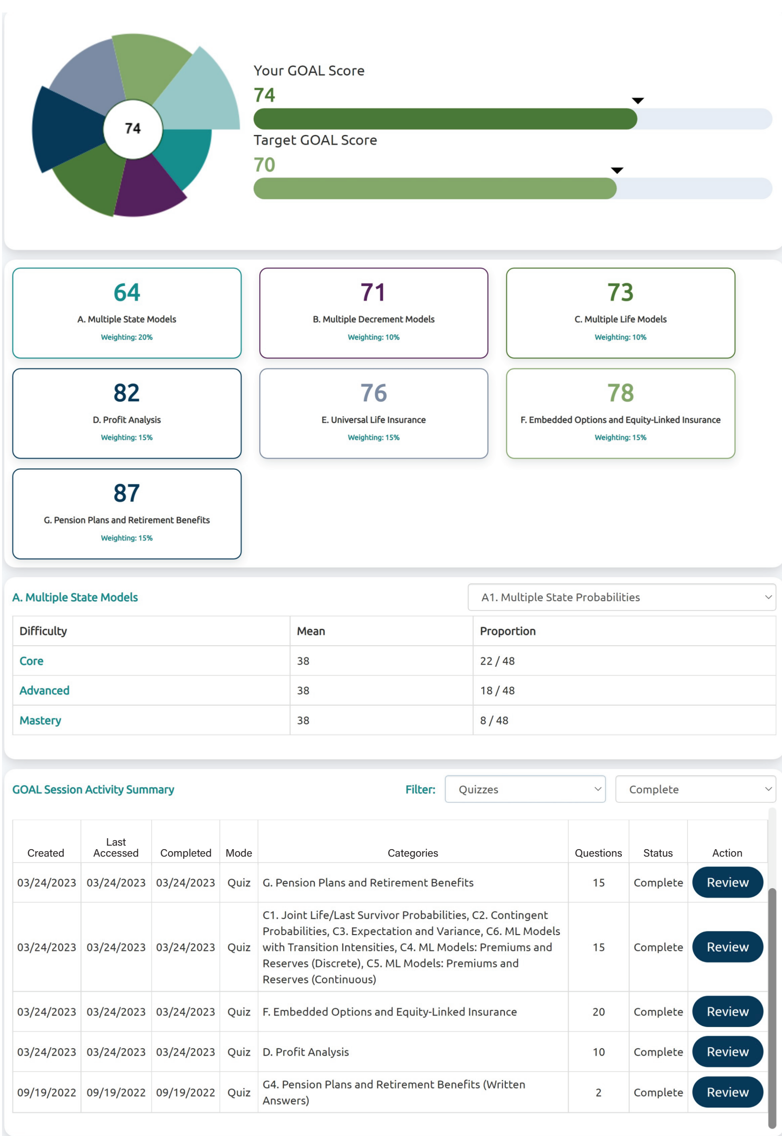

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Contents

Preface

Thank you for choosing ACTEX.

In 2022, the SoA launched Exam ALTAM (Advanced Long-Term Actuarial Mathematics). This exam replaces part of Exam LTAM that was offered in the past and includes some topics that were not examined in the SoA curriculum before.

To help you prepare for Exam ALTAM, ACTEX developed this brand new study manual, written by Professors Johnny Li and Andrew Ng who have deep knowledge in the exam topics. They have published several articles in top-tier academic journals on variable annuity guarantees – a topic that is newly introduced to the SoA curriculum through Exam ALTAM.

The learning outcomes stated in the syllabus of Exam ALTAM require candidates to be able to interpret a lot of actuarial concepts. This skill is drilled extensively in our practice problems, which often ask you to interpret a certain actuarial formula or to explain your calculation. Also, in Exam ALTAM you may be asked to define or describe a certain product, model or terminology. To help you prepare for this type of questions, we provide you with the definitions and descriptions of various products and terminologies throughout the study manual.

Proofs and derivations are another key challenge. In Exam ALTAM, you are highly likely to be asked to prove or derive something. You are expected to know, for example, how to derive the Kolmogorov forward differential equations for a certain transition probability. In this new study manual, we do teach (and drill) you how to prove or derive important formulas. This is in stark contrast to some other exam prep products in which proofs and derivations are downplayed, if not omitted.

Besides the topics specified in the exam syllabus, you also need to know a range of numerical techniques, for example, Euler's method and Simpson's rule, in order to succeed. We know that you may not have even seen these techniques before, so we have prepared a special chapter (Appendix 1) to teach you all of the numerical techniques required for Exam ALTAM. In addition, whenever a numerical technique is used, we clearly point out which technique it is, letting you follow our examples and exercises more easily.

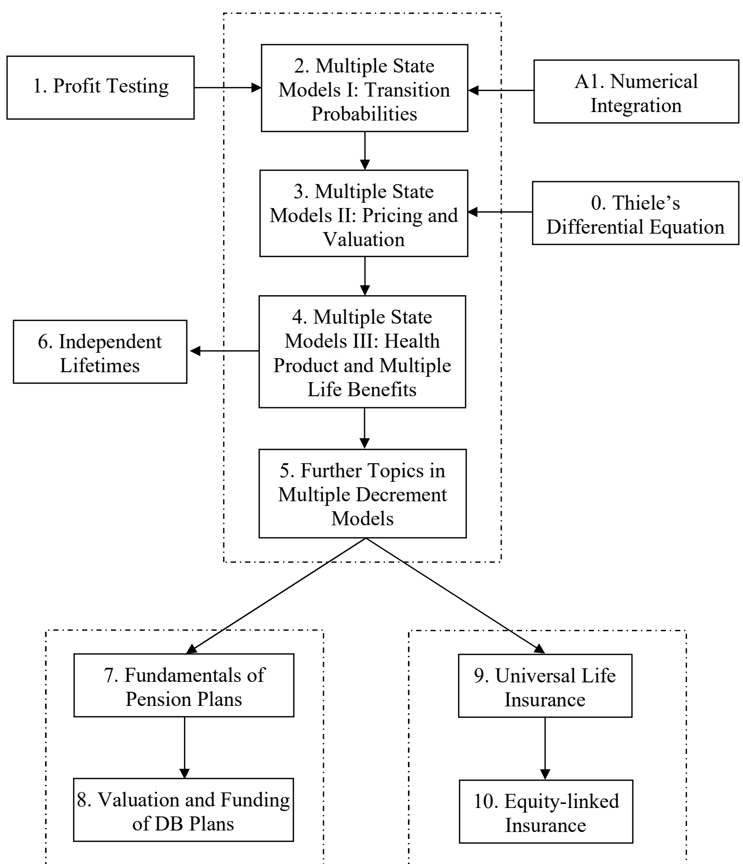
Starting April 2024, every ATLAM exam paper contains a spreadsheet-based question, which must be answered using Excel. All mock exams provided in this study manual includes questions of this type. Solutions in Excel format are provided.

Other distinguishing features of this study manual include:

- We use graphics extensively. Graphical illustrations are probably the most effective way to explain formulas involved in Exam ALTAM. The extensive use of graphics can also help you remember various concepts and equations.
- A sleek layout is used. The font size and spacing are chosen to let you feel more comfortable in reading. Important equations are displayed in eye-catching boxes.
- A unique pedagogical flow is adopted. For instance, in contrast to Actuarial Mathematics for Life Contingent Risks, we cover discrete-time Markov Chains before continuous-time Markov Chains (which involves a lot of calculus). We believe that it is more effective to learn easier topics before difficult ones.

- Rather than splitting the manual into tiny units, each of which tells you a couple of formulas only, we have carefully grouped the exam topics into 10 chapters and 2 appendices. Such a grouping allows you to more easily identify the linkages between different concepts, which are essential for your success as multiple learning outcomes can be examined in one single exam question.
- Instead of giving you a long list of formulas, we point out which formulas are the most important. Having read this study manual, you will be able to identify the formulas you must remember and the formulas that are just variants of the key ones.
- We do not want to overwhelm you with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and integrated into the practice problems.
- We explain multiple-state models in great depth. A solid understanding of multiple-state models is crucially important, because many of the learning objectives in Exam ALTAM are related to multiple-state models.
- We teach you how to make tedious retiree health benefit calculations more manageable by using a tabular approach. Also, whenever possible, multiple methods (direct methods and computationally efficient algorithms) are presented.
- We group materials related to profit testing into one single chapter. This arrangement will help you more easily understand profit testing concepts and techniques.

The following flowchart shows how different chapters of this manual are connected to one another.



You should first study Chapters 1 to 4 in order. These four chapters will build you a solid foundation. You may continue with the following three paths in any order you wish: (i) Chapter 6, (ii) Chapters 5, 7 and 8, and (iii) Chapters 5, 9 and 10. Immediately after reading a chapter, do all practice problems we provide for that chapter. Make sure that you understand every single practice problem. Finally, work on the mock exams.

Before you begin your study, please download and read the exam syllabus from the SoA's website:

https://www.soa.org/education/exam-req/edu-exam-altam/

You should also check the exam home page periodically for updates, corrections or notices.

If you find a possible error in this manual, please let us know at the "Contact Us" link on the ACTEX homepage (www.actexlearning.com). Any confirmed errata will be posted on the ACTEX website under the "Errata" link.

Enjoy your study!

Chapter 1

Profit Testing

Objectives

(1) To calculate emerging surplus, profit vector and profit signature

(2) To calculate profitability measures of traditional insurance policies

1.1 Actual and Expected Profit

Suppose that a profit is loaded into the premium either implicitly or explicitly. Then we can calculate expected and actual profit over each policy year. In Exam ALTAM, the following definitions are given:

For a time period where all cash flows occur only at the beginning and end of the time period:

Profit for the time period occurs at the end of the time period and is (a) minus (b) where:

(a) is the accumulated value of the sum of the policy value (aka reserve) at the end of the previous period and the cash flows that occur at the beginning of the year; and

(b) is the sum of the values of the policy value at the end of the period and the cash outflows that occur at the end of the period.

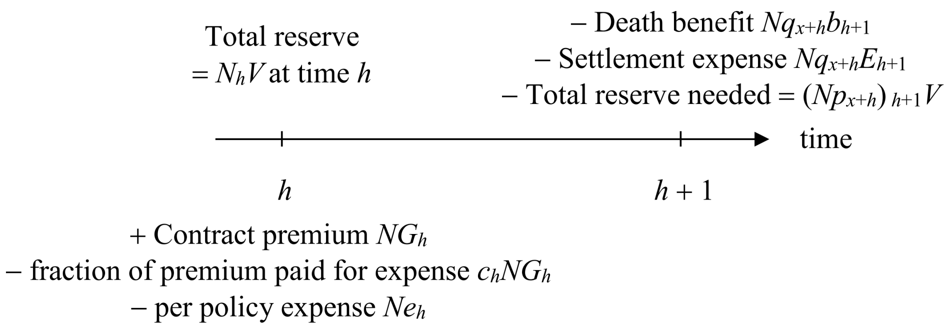
Expenses at the inception of a contract may be classified as negative profit at time 0 or may be part of period 1 cash flow and included in the profit calculation for period 1. Any initial expenses that are not part of period 1 cash flow will be identified in the question as pre-contract expenses. If a reserve is to be established at time 0, before the first premium is received, it would be part of the time 0 profit. Any such reserve will be identified in the question.

Expected profit is the profit calculated using the gross premium, expected cash flows at the beginning and end of the period, and accumulating beginning of year values. The assumptions for expected cash flows and the expected interest rate may or may not be the assumptions in the reserve model.

Actual profit is calculated using the gross premium, actual cash flows at the beginning and end of the period, and accumulating at the beginning of year values using the actual investment rate earned during the period.

Gain is the actual profit minus the expected profit for the period. **Gain by source** is the gain calculated where the effect of the difference between the observe values and the expected values in the profit calculations from one source is reflected, while the differences for the other sources are not.

We now decode these definitions. Firstly, in the calculation of profit, the actual premium charged should be used. Let's look at the $(h+1)$th policy year, where $h > 1$. At the beginning of the $(h+1)$th policy year, there are N in force policies, and the insurance company holds a policy value of ${}_hV$ for each policy. The policy value can be a net premium policy value, a gross premium policy value or other kinds of policy value, depending on the situation given.



- The expected profit that emerges in the $(h+1)$th policy year, under a specific basis, is

$$\mathrm{Pr}_{h+1} = N[{}_hV + G_h(1-c_h) - e_h](1+i_{h+1}) - N[(b_{h+1}+E_{h+1})q_{x+h} + p_{x+h}\ {}_{h+1}V].$$

Notice that the specific basis used can be different from that of used to compute the policy values. If this valuation basis, the premium basis and the basis for which Pr_{h+1} is calculated are the same and G_h is determined from the equivalence principle, then $\mathrm{Pr}_{h+1} = 0$. However, this may not be the case, and an expected profit or loss will then emerge.

- For actual profit, we denote the actual interest rate, percentage of premium expense, per policy expense, death rate, and settlement expenses by $\hat{i}_{h+1}, \hat{c}_h, \hat{e}_h, \hat{q}_{x+h}, \hat{E}_{h+1}$, respectively. The actual profit can then be expressed as

$$\widehat{\mathrm{Pr}}_{h+1} = N[{}_hV + G_h(1-\hat{c}_h) - \hat{e}_h](1+\hat{i}_{h+1}) - N[(b_{h+1}+\hat{E}_{h+1})\hat{q}_{x+h} + \hat{p}_{x+h}\ {}_{h+1}V].$$

When a profit is loaded into the premium explicitly, or when the actual experience of mortality, expense and interest turn out to be better than assumed in the valuation basis, then every year there is an assumed amount of profit that emerges from the policy.

The gain is the actual profit less expected profit:

$$\mathrm{Gain}_{h+1} = \widehat{\mathrm{Pr}}_{h+1} - \mathrm{Pr}_{h+1}$$

Remember that in the discussion above we let $h > 1$. Why is the 1st policy year different? The initial expense at time 0 may include commissions, underwriting fees, legal fees, acquisition costs, etc. As a result, e_0 (or $\hat{e}_0$) would typically be very large and this would drag down the profit that emerged during the first policy year, making it out of line with profits in later policy years. To avoid this happening, some initial expenses may not be incorporated in Pr_1 (or $\widehat{\mathrm{Pr}}_1$) but are incorporated as a negative profit at time 0. We shall illustrate this point more clearly in the next section.

Example 1.1. Consider a 4-year annual level premium special endowment insurance on (40). The death benefit is 1000, payable at the end of the year of death if death occurs within 4 years. If the life survives 4 years, 1200 is payable 4 years after the issuance of the policy. The premium charged is 350, and the gross premium policy values are ${}_1V^g = 182.2313$ and ${}_2V^g = 470.5737$.

The actual experience at the end of year 1, 2 and 3 are as follows:

- the settlement expense is only 4 dollars at the end of the first 3 years,
- the renewal expense is 2.5 per policy plus 5% of each premium,
- at the start there were 500 policies, and the number of deaths during the first, second and the third year are 25, 48 and 40, respectively,
- the interest rates during the first, the second and the third year are 6%, 5% and 6.5%, respectively.

Expected profit is calculated based on the following:

(i) Mortality:

$$q_{40} = 0.08, \quad q_{41} = 0.09, \quad q_{42} = 0.10, \quad q_{43} = 0.11$$

(ii) Expenses: Initial expenses and renewal expenses (payable at the beginning of each policy year)

Year	Per policy	Percentage of premium
1st year	10	25%
Renewal years	2	5%

Settlement expense is 5, payable at the end of the year for the first 3 years. There is no settlement expense in the fourth year.

(iii) Interest: $i = 0.06$

Calculate the gain in the second policy year.

Solution: The total number of survivors at the beginning of the second policy year is $N = 500 - 25 = 475$.

The actual profit is

$$475[182.2313 + 350(1 - 0.05) - 2.5] \times (1 + 0.05) - (48 \times 1004 + 427 \times 470.5737) = 6{,}348.3910.$$

The expected profit is

$$\begin{aligned} &475[182.2313 + 350(1 - 0.05) - 2] \times (1 + 0.06) - 475 \times (0.09 \times 1005 + 0.91 \times 470.5737) \\ &= 11{,}790.9777. \end{aligned}$$

The gain is $-5{,}442.5867$. □

Gain by Source

We can further break down the gain into the gains from mortality, from interest and from expenses. This decomposition, however, would depend on the order of the split. In the following discussion, we split the actual gain in the following order:

interest, expenses, mortality

Interest component:

In this calculation, we assume that expenses and mortality follow the assumptions. Interest only affects the cash flows that happen at the beginning of the period.

$$\begin{aligned}\mathrm{Pr}^i_{h+1} &= N[{}_hV + G_h(1-c_h) - e_h](1+\hat{i}_{h+1}) - N[{}_hV + G_h(1-c_h) - e_h](1+i_{h+1})\\ &= N[{}_hV + G_h(1-c_h) - e_h](\hat{i}_{h+1} - i_{h+1})\end{aligned}$$

If the realized interest rate is greater than that assumed, the interest component would be positive.

Expense component:

In this calculation, we assume that mortality follows the assumption, but interest follows the experience.

$$\begin{aligned}\mathrm{Pr}^e_{h+1} = N[{}_hV + G_h(1-\hat{c}_h) - \hat{e}_h](1+hati_{h+1}) - N[(b_{h+1} + \hat{E}_{h+1})q_{x+h} + p_{x+h}\ {}_{h+1}V]\\ - N[{}_hV + G_h(1-c_h) - e_h](1+\hat{i}_{h+1}) + N[(b_{h+1} + E_{h+1})q_{x+h} + p_{x+h}\ {}_{h+1}V]\\ = -N[G_h(\hat{c}_h - c_h) + (\hat{e}_h - e_h)](1+\hat{i}_{h+1}) - N(\hat{E}_{h+1} - E_{h+1})q_{x+h}\end{aligned}$$

If the realized sum of expenses is greater than that assumed, the expense component would be negative.

Mortality component:

In this calculation, we assume that both interest and expenses follow the experience:

$$\begin{aligned}\mathrm{Pr}^d_{h+1} &= -N[(b_{h+1} + \hat{E}_{h+1})\hat{q}_{x+h} + \hat{p}_{x+h}\ {}_{h+1}V] + N[(b_{h+1} + \hat{E}_{h+1})q_{x+h} + p_{x+h}\ {}_{h+1}V]\\ &= N(b_{h+1} + \hat{E}_{h+1})(q_{x+h} - \hat{q}_{x+h}) + N(p_{x+h} - \hat{p}_{x+h})\ {}_{h+1}V\\ &= -N(b_{h+1} + \hat{E}_{h+1} - {}_{h+1}V)(\hat{q}_{x+h} - q_{x+h})\end{aligned}$$

In the above, the final expression is always of the form "actual − expected". This allows you to attach the correct sign (+ or −) to the expression.

The total gain is

$$\widehat{\mathrm{Pr}}_{h+1} - \mathrm{Pr}_{h+1} = \mathrm{Pr}^i_{h+1} + \mathrm{Pr}^e_{h+1} + \mathrm{Pr}^d_{h+1}\,.$$

Example 1.2. Decompose the gain in the second year in Example 1.1 into profit / loss attribute to interest, expense and mortality.

Solution: The interest component is

$$475[182.2313 + 350(1 - 0.05) - 2] \times (0.05 - 0.06) = -2{,}435.4736.$$

The expense component is

$$-475 \times 0.5 \times 1.05 + 475 \times 0.09 \times 1 = -206.625.$$

The actual death rate is $48/475 = 0.10105263$.
The mortality component is

$$-475 \times (1004 - 470.5737) \times (0.10105263 - 0.09) = -2{,}800.4877.$$

The sum of the three components is $-5{,}442.586$. The slight difference is due to rounding. □

You can decompose the gain in any order. The rule is that at each step you assume that factors not yet considered are as specified in the assumption, whereas factors already considered are as actually occurred.

1.2 Profit Vector and Profit Signature

In the previous section the profit over a period was defined as

- the accumulated value (using the period effective interest rate) of the sum of the reserve at the end of the previous period and the cash flows that occur at the beginning of the period

less

- the sum of the value of the reserve at the end of the period and the cash outflows that occur at the end of the period.

Expected profit is the profit calculated using the values that were anticipated for the next time period prior to the start of that time period, while the actual profit is the profit calculated using the values that were observed during the time period.

Consider a fully discrete policy, with N policies in force at the beginning of the $(h+1)$ th policy year. The insurance company holds a reserve of ${}_hV$ for each policy.

The **expected** profit that emerges in the $(h+1)$th policy year is then

$$\mathrm{Pr}_{h+1} = N[{}_hV + G_h(1 - c_h) - e_h](1 + i_{h+1}) - N[(b_{h+1} + E_{h+1})q_{x+h} + p_{x+h}\ {}_{h+1}V.$$

Let us make a few observations concerning the expression above.

(1) The premium G_h is calculated under some specific assumptions on mortality, expenses and interest. The assumptions are collectively called the premium basis.

(2) The reserves ${}_hV$ and ${}_{h+1}V$ are the actual amount of money held by the insurer. Recall from Exam FAM-L that the term "policy value" and reserves are sometimes used interchangeably. However, in this chapter we make the fine distinction that reserves are generally no less than policy values. Of course we can still set reserve to be equal to policy

value. But it can also be determined in an arbitrary manner. If we use a set of mortality, expenses and interest to determine the reserves, then the set of assumptions is called a reserve basis.

(3) In the calculation of profit, we can use another set of assumptions on mortality, expenses and interest. This basis is called the profit test basis.

Since we would handle the effect of N later, we now let $N = 1$ for every policy year. This means that Pr_{h+1} is the profit for a single policy in force at time h. So, the profit can be written as

$$\begin{aligned}
\text{Pr}_{h+1} &= [{}_hV + G_h(1-c_h) - e_h](1+i_{h+1}) - [(b_{h+1}+E_{h+1})q_{x+h} + p_{x+h}\ {}_{h+1}V] \\
&= [G_h(1-c_h) - e_h](1+i_{h+1}) - (b_{h+1}+E_{h+1})q_{x+h} + \underbrace{(1+i_{h+1})\,{}_hV - p_{x+h}\ {}_{h+1}V}_{\text{change in reserve in year } h+1}
\end{aligned}$$

for $h = 0, 1, 2, \ldots$. For the first policy year, c_hG_h and e_h may not be included if they are treated as acquisition costs. The general rule is: pre-contract expenses would be accounted for by Pr_0, and other expenses would be incorporated into Pr_1. So, $\text{Pr}_0 = -E_0 - {}_0V$, where E_0 is the pre-contract expense and ${}_0V$ is the time-0 reserve (which is zero if net premium or gross premium policy value is used to determine the reserve).

Sometimes a special treatment has to be made for the final profit for an endowment policy. Suppose the survival benefit is S and the final settlement expense is E_n. Then

$$\begin{aligned}
\text{Pr}_n &= [{}_{n-1}V + G_{n-1}(1-c_{n-1}) - e_{n-1}](1+i_n) - (S+E_n) \\
&= [G_{n-1}(1-c_{n-1}) - e_{n-1}](1+i_n) - (S+E_n) + (1+i_n)\,{}_{n-1}V
\end{aligned}$$

and hence the change in reserve is simply $(1+i_n)\,{}_{n-1}V$ (though ${}_nV = S$, the convention is to ignore it),

The following profit calculation explains why reserve is generally needed for proper risk management:

Example 1.3. Consider a 4-year annual level premium term life insurance issued to a life aged 40. The death benefit is 10000, payable at the end of the year of death.

The premium basis is as follows:

(i) Mortality: $q_{40} = 0.08$, $q_{41} = 0.09$, $q_{42} = 0.10$, $q_{43} = 0.11$.

(ii) Expenses: Pre-contract expenses: 40 plus 25% of the first premium
Renewal and other expenses: 5% of the premiums, including the first at the beginning of each period.
No settlement expenses.

(iii) Interest: 4% annual effective during the first two years, 5% during the next two years.

(a) Calculate the gross premium.

(b) Calculate the profits at time 0, 1, 2, 3 and 4, assuming that the insurance company holds no reserves for the term policy and that the profit test basis is the same as the premium basis, except that the interest on insurer's asset is always 5%.

Solution:

(a) To compute various insurance and annuity functions, we first construct the following table:

k	q_{40+k}	${}_kp_{40}$	${}_{k\|}q_{40} = {}_kp_{40}q_{40+k}$	$v(k)$
0	0.0	1	0.08	1
1	0.09	0.92	0.0828	1.04^{-1}
2	0.10	0.8372	0.08372	1.04^{-2}
3	0.11	0.75348	0.082883	$1.04^{-2}1.05^{-1}$
4		0.670597		$1.04^{-2}1.05^{-2}$

As a result, $\ddot{a}_{40:\overline{4}|} = \sum_{k=0}^{3} v(k)_k p_{40} = 3.3221154$, $A^{1}_{40:\overline{4}|} = \sum_{k=0}^{3} v(k+1)_{k|}q_{40} = 0.2967$.

By the equivalence principle,

$$10000A^{1}_{40:\overline{4}|} + 40 + 0.25G = 0.95G\ddot{a}_{40:\overline{4}|}.$$

On solving, we get $G = 1034.7523$.

(b) $\mathrm{Pr}_0 = -40 - 0.25G = -298.688$

$\mathrm{Pr}_1 = 0.95G \times 1.05 - 10000q_{40} = 232.165$

$\mathrm{Pr}_2 = 0.95G \times 1.05 - 10000q_{41} = 132.165$

$\mathrm{Pr}_3 = 0.95G \times 1.05 - 10000q_{42} = 32.165$

$\mathrm{Pr}_4 = 0.95G \times 1.05 - 10000q_{43} = -67.835$ □

The example above shows that the profits decrease steadily and become negative during the final policy year. This is a general phenomenon. The reason behind this is simple: the premium is level, but the expected death claim (or survival benefit, if there is any) usually increases as time goes on. If the insurance company distributes the profits in early years as dividends to shareholders but does not set a reserve, then in later years there would not be enough cash to pay benefits.

Now suppose the insurer holds net premium policy values and that the mortality and interest assumptions under the net premium policy value basis are the same as those under the premium basis. The net premium is

$$0000P^{1}_{40:\overline{4}|} = 10000\frac{A^{1}_{40:\overline{4}|}}{\ddot{a}_{40:\overline{4}|}} = 893.105.$$

Then by using Fackler's accumulation formula, starting from ${}_0V = 0$, we have

$${}_1V = \frac{893.105 \times 1.04 - 10000 \times 0.08}{0.92} = 140.0317,$$

$${}_2V = \frac{(893.105 + 140.0317) \times 1.04 - 10000 \times 0.09}{0.91} = 191.7166,$$

$${}_3V = \frac{(893.105 + 191.7166) \times 1.05 - 10000 \times 0.1}{0.9} = 154.5141,$$

and ${}_4V = 0$.

The profits at time 1, 2, 3 and 4 become:

$$\begin{aligned}
\text{Pr}_1 &= 232.165 + 1.05\,{}_0V - 0.92\,{}_1V = 103.335\\
\text{Pr}_2 &= 132.165 + 1.05\,{}_1V - 0.91\,{}_2V = 104.736\\
\text{Pr}_3 &= 32.165 + 1.05\,{}_2V - 0.90\,{}_3V = 94.405\\
\text{Pr}_4 &= -67.835 + 1.05\,{}_3V - 0.89\,{}_4V = 94.405
\end{aligned}$$

So you can see that the profits become steady and non-negative.

Now we are in a position to define profit vector and profit signature:

Profit Vector and Profit Signature

The column vector

$$\mathbf{Pr} = (\text{Pr}_0,\ \text{Pr}_1,\ \text{Pr}_2,\ \text{Pr}_3,\ \ldots)'$$

is called the **profit vector** for the contract.

The column vector

$$\mathbf{\Pi} = (\Pi_0,\ \Pi_1,\ \Pi_2,\ \Pi_3,\ \ldots)' = (\text{Pr}_0,\ \text{Pr}_1,\ p_x\text{Pr}_2,\ {}_2p_x\text{Pr}_3,\ \ldots)'$$

is called the **profit signature** for the contract.

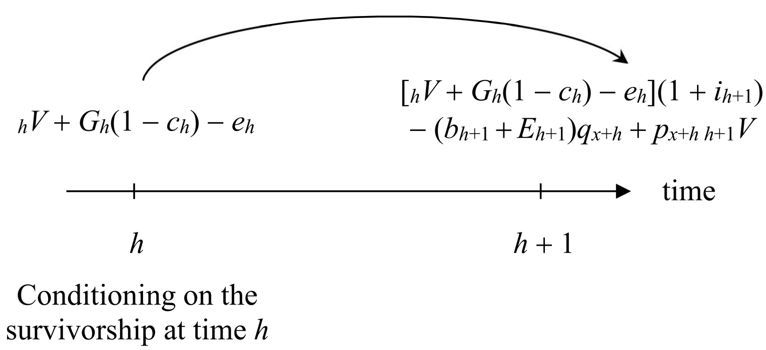
(Recall that A' is the transpose of matrix A.)

The profit signature can be thought of as the expected (undiscounted) emerging profit at issue of the policy.

As a simple illustration, in Example 1.3, the profit signature is

$$\begin{aligned}
\mathbf{\Pi} &= (-298.688,\ 103.335,\ 0.92 \times 104.736,\ 0.8372 \times 94.405,\ 0.75348 \times 94.405)'\\
&= (-298.688,\ 103.335,\ 96.357,\ 79.035,\ 71.132)'.
\end{aligned}$$

Notice that while the notation $\text{Pr}_{h+1} (h \geq 0)$ suggests that it is a cash flow at time $h+1$, so that its appropriate discount factor seems to be ${}_{h+1}E_x = v(h+1)\,{}_{h+1}p_x$, this is not the case! The correct discount factor turns out to be $v(h+1)\,{}_hp_x$. Look carefully at the structure of Pr_{h+1}:



We have conditioned on survivorship at time h when defining Pr_{h+1}, and q_{x+h} and p_{x+h} have already appeared when we calculate the expected loss at the end of the period. If we incorrectly multiply Pr_{h+1} by ${}_{h+1}p_x = {}_hp_x p_{x+h}$ when we compute its expectation at time 0, we would have double counted the probability in $(h, h+1)$.

Example 1.4. Consider a 3-year fully discrete term insurance of 1000 on (40). You are given:

(i) Premium basis:

Mortality: $q_{40} = 0.08$, $q_{41} = 0.09$, $q_{42} = 0.10$
Expenses:
Initial expenses and renewal expenses (payable at the beginning of each policy year)

Year	Per policy	Percentage of premium
1^{st} year	10	25%
Renewal years	2	5%

Claim expense is 5
Interest: 6% annual effective

(ii) Net Premium Policy Value basis:

Mortality: $q_{40} = 0.10$, $q_{41} = 0.10$, $q_{42} = 0.11$
Interest: 5.5% annual effective

The premium for this policy is 102.42811, and the net premium policy values are

$$_0V = 0, \quad _1V = 3.133132, \quad _2V = 6.8058589, \quad _3V = 0$$

Suppose that the profit test basis is the same as the premium basis, except that the settlement expense is 4. Treat all expenses incurred at the beginning the first policy year as pre-contract expenses at time 0. Reserves are set equal to policy values.

Calculate the profit vector and profit signature for the contract.

Solution:

$\text{Pr}_0 =$ negative of sum of initial expenses $= -10 - 0.25G = -35.6070275$

$\text{Pr}_1 = G \times 1.06 - (0.08 \times 1004 + 0.92\,_1V) = 25.3713152$

$\text{Pr}_2 = (_1V + 0.95G - 2) \times 1.06 - (0.09 \times 1004 + 0.91\,_2V) = 7.79289509$

$\text{Pr}_3 = (_2V + 0.95G - 2) \times 1.06 - (0.1 \times 1004) = 7.8393172$

So the profit vector is

$$\mathbf{Pr} = (-35.6070275,\ 25.3713152,\ 7.7928951,\ 7.8393172)'.$$

Noting that $q_{40} = 0.08$, $q_{41} = 0.09$, $q_{42} = 0.10$ under the profit test basis, the profit signature is

$$\begin{aligned}\mathbf{\Pi} &= (-35.6070275,\ 25.3713152,\ 7.7928951 \times 0.92,\ 7.8393172 \times 0.92 \times 0.91)' \\ &= (-35.6070275,\ 25.3713152,\ 7.1694635,\ 6.5630764)'.\end{aligned}$$

□

Extension to Policies with Continuous Benefit

In the previous example we have assumed that benefits are payable at the end of the year. What if benefits are payable immediately on death? In this case, a crude approximation (similar to the claims acceleration approach in Exam FAM-L) is to assume that deaths occur in the middle of the year, and accumulate the expected death benefits to the end of the period by multiplying it by $(1+i_{h+1})^{1/2}$. This means we use

$$\begin{aligned}\Pr_{h+1} =& [{}_hV + G_h(1-c_h) - e_h](1+i_{h+1}) \\ & - [(1+i_{h+1})^{1/2}(b_{h+0.5} + E_{h+0.5})q_{x+h} + p_{x+h}\ {}_{h+1}V].\end{aligned}$$

Example 1.5. Consider a 1-year term insurance on (58). \$100,000 is payable at the moment of death, while the quarterly premium of 200 is throughout the term. The policyholder is subject to Makeham's law of mortality with parameters $A = 0.0002$, $B = 3 \times 10^{-6}$, and $c = 1.12$.

Under the policy value basis,

- expenses are ignored,
- the mortality rates of a 60-year old is used,
- the annual effective interest rate is 6%.

This results in the following quarter-end reserves:

$${}_{0.25}V = 2.982944,\ {}_{0.5}V = 4.035478,\ {}_{0.75}V = 3.071047$$

Under the profit test basis,

- initial expenses are 300 plus 50% of the first quarterly premium; renewal expenses are 10% of each subsequent premium; settlement expense is 0.
- the mortality rates of a 58-year old is used.
- the annual effective interest rate is 8%.

Assuming deaths occur at the middle of each quarter, calculate the profit signature.

Solution: First we find all survival probabilities required in the calculation. It is easy to verify that

x	58	58.25	58.5	58.75
${}_{0.25}p_x$	0.999405833	0.999390199	0.999374117	0.999357572

We can then calculate the following:

t	0.25	0.5	0.75
${}_tp_{58}$	0.999405833	0.998796394	0.998171264

Obviously, $\Pr_0 = -300 - 0.5 \times 200 = -400$.

Now we can calculate the profit emerging at each quarter end.

$$\begin{aligned}\mathrm{Pr}_{0.25} &= 200 \times 1.08^{1/4} - {}_{0.25}p_{58} \times {}_{0.25}V - {}_{0.25}q_{58} \times 100{,}000 \times 1.08^{1/8} \\ &= 140.913083.\end{aligned}$$

(Note: if the death benefit is payable at the end of each quarter, then

$$\begin{aligned}\mathrm{Pr}_{0.25} &= 200 \times 1.08^{1/4} - {}_{0.25}p_{58} \times {}_{0.25}V - {}_{0.25}q_{58} \times 100{,}000 \\ &= 141.4874377)\end{aligned}$$

$$\begin{aligned}\mathrm{Pr}_{0.5} &= ({}_{0.25}V + 200 \times 0.9) \times 1.08^{1/4} - {}_{0.25}p_{58.25} \times {}_{0.5}V - {}_{0.25}q_{58.25} \times 100{,}000 \times 1.08^{1/8} \\ &= 120.935086\end{aligned}$$

$$\begin{aligned}\mathrm{Pr}_{0.75} &= ({}_{0.5}V + 200 \times 0.9) \times 1.08^{1/4} - {}_{0.25}p_{58.5} \times {}_{0.75}V - {}_{0.25}q_{58.5} \times 100{,}000 \times 1.08^{1/8} \\ &= 121.348214\end{aligned}$$

$$\begin{aligned}\mathrm{Pr}_{1} &= ({}_{0.75}V + 200 \times 0.9) \times 1.08^{1/4} - {}_{0.25}q_{58.75} \times 100{,}000 \times 1.08^{1/8} \\ &= 121.763679\end{aligned}$$

The profit signature is

$$\begin{aligned}&(\mathrm{Pr}_0 - 400,\ \mathrm{Pr}_{0.25},\ \mathrm{Pr}_{0.5} \times 0.999405833,\ \mathrm{Pr}_{0.75} \times 0.998796394,\ \mathrm{Pr}_1 \times 0.998171264)' \\ &= (-400,\ 140.913083,\ 120.863230,\ 121.202159,\ 121.541005)'.\end{aligned}$$

□

1.3 Profit Measures

After we have calculated the profit signature, we can compute a variety of profit measures under the framework of discounted cash flow analysis and capital budgeting. All these measures are generalizations of profit measures for deterministic cash flows. So let's have a brief review of these profit measures in the simplest deterministic setting.

Consider an investment that leads to cash flows of $C_0, C_1, C_2, \ldots, C_n$ at times $0, 1, 2, \ldots, n$.

1. Net present value (NPV)

   Given a **risk discount rate** r (This is not the effective rate of discount d. We call it a risk discount rate, just because it is used to discount (contingent) cash flows.), the NPV of the cash flow stream is

   $$\mathrm{NPV}(r) = \sum_{k=0}^{n} \frac{C_k}{(1+r)^k}.$$

   If the NPV is positive, then the investment is deemed profitable.

   The SOA also introduces the term **partial net present value**

   $$\mathrm{NPV}(r; m) = \sum_{k=0}^{m} \frac{C_k}{(1+r)^k}.$$

2. Internal rate of return (IRR)

   The internal rate of return is the zero of the equation

$$\mathrm{NPV}(r) = \sum_{k=0}^{n} \frac{C_k}{(1+r)^k} = 0.$$

   If $\{C_k\}$ changes sign only once, then the IRR is unique. Otherwise, there may be multiple IRRs, or no IRR (this is the case when all cash flows carry the same sign).

   Very frequently an investor or a company would specify a minimum discount rate (or **hurdle rate**) which is also known as the "required rate of return." If the IRR exceeds the hurdle rate, then the investment is deemed profitable. This rule works well if C_k's are initially negative, and then become positive.

3. Discounted payback period (DPP)

   Given the hurdle rate r, the discounted payback period (also known as the break-even period) is the smallest value of m such that

$$\mathrm{NPV}(r;m) = \sum_{k=0}^{m} \frac{C_k}{(1+r)^k} \geq 0.$$

   Loosely speaking, DPP is the time until the investment starts to make a profit. If no such m exists, then the investment never pays back.

4. Profit margin

   Profit margin is the NPV of net cash flows as a percentage of the NPV of the revenues. Suppose that the revenue cash flows are $R_0, R_1, R_2, \ldots, R_n$. Then the profit margin is

$$\frac{\mathrm{NPV}(r)}{\displaystyle\sum_{k=0}^{n} \frac{R_k}{(1+r)^k}}.$$

For life insurance policies, we replace C_k by Π_k. The NPV becomes the expected present value of the profits at issue. For profit margin, G_k is the premium received at time k. Since premiums are life-contingent,

$$\text{Profit margin} = \sum_{k=0}^{n} \frac{\Pi_k}{(1+r)^k} \Bigg/ \sum_{k=0}^{n-1} \frac{G_k\,{}_kp_x}{(1+r)^k}.$$

Here the survival probabilities ${}_kp_x$ come from the profit test basis. The risk discount rate, r, is not necessarily the same as the interest rate specified in the profit test basis for the calculation of the profit vector.

Another measure useful for life insurance policies is the NPV as a proportion of the acquisition costs. This equals

$$\frac{1}{E_0} \sum_{k=0}^{n} \frac{\Pi_k}{(1+r)^k}.$$

<u>Miscellaneous Information: Calculator Trick</u>

To find NPV when n is large, you can use BA II plus. Suppose that you have the following cash flow stream:

$$C_0 = -450,\ C_1 = 25,\ C_2 = 320,\ C_3 = 160$$

and $r = 10\%$. The following series of keystrokes calculates the NPV.

First clear the cash flow worksheet by pressing [CF] [2nd] [CLR WORK].

Step 1: Press [CF], the display should read CF0 =

Step 2: Press 450 [+/−] [ENTER] [↓], the display should read C01 =

Step 3: Press 25 [ENTER] [↓] [↓] (the 2nd arrow means we ignore F01), the display should read C02 =

Step 4: Press 320 [ENTER] [↓] [↓], the display should read C03 =

Step 5: Press 160 [ENTER].

Step 6: Press [NPV]. The display should read I = 0. Press 10 [ENTER] [↓] [CPT].

The output is 42.59955.

(In case your output is only 2 d.p. and you want to see more number of decimal places, you can adjust that with the following keystrokes: [2nd] [FORMAT](which would then give DEC = 2.00 along with the word ENTER). Press the number of decimal places you want followed by [ENTER].)

To find the IRR when $n \geq 3$, you need to use BA II plus. Follow Steps 1 to 5 above to enter the cash flow stream. Then use:

Step 6: Press [IRR] [CPT].

The display should read 5.24. The IRR is 5.2351%. Notice that if there are multiple IRR, then BA II plus would only report the one that is closest to zero.

Example 1.6. Consider the set up in Example 1.4. Calculate the profit measures introduced above. Use a risk discount rate of 5% effective per annum.

Solution: We have $\Pi = (-35.6070275, 25.3713152, 7.1694635, 6.5630764)'$.

The expected present value of the profits at issue is

$$\text{NPV}(0.05) = -35.6070275 + \frac{25.3713152}{1.05} + \frac{7.1697635}{1.05^2} + \frac{6.5630764}{1.05^3} = 0.728477.$$

By using BA II plus, the IRR is found to be 6.44%.

To calculate the discounted payback period,

m	1	2	3
NPV$(5\%; m)$	-11.443870	-4.94096	0.728477

So the discounted payback period is 3.

To find the profit margin, note that $G = 102.42811$, and that

$$\sum_{k=0}^{2} \frac{G_k {}_k p_x}{(1+r)^k} = G\left(1 + \frac{0.92}{1.05} + \frac{0.92 \times 0.91}{1.05^2}\right) = 269.95497.$$

The profit margin is $0.728477/269.95497 = 0.27\%$.

The NPV as a percentage of acquisition costs is $0.728477/35.6070275 = 2.05\%$.

□

Reserves have a huge impact on profit measures. To investigate the effect, consider an n-year term insurance. Let i be the return on asset under the profit test basis. We separate the profit into terms involving and not involving reserves:

The profit signature is computed using $\Pi_{h+1} = {}_h p_x \mathrm{Pr}_{h+1}$ (with the exception of the first item $\mathrm{Pr}_0 =$ pre-contract expense $- {}_0V$. As a result, the NPV under a hurdle rate r is

$$\begin{aligned}
&-E_0 - {}_0V + \sum_{h=1}^{n} \frac{\Pi_h}{(1+r)^h} \\
&= -E_0 - {}_0V + \frac{1}{1+r} \sum_{h=0}^{n-1} \frac{\{[G_h(1-c_h) - e_h](1+i) - (b_{h+1} + E_{h+1}) q_{x+h}\} {}_h p_x}{(1+r)^h} \\
&\quad + \sum_{h=0}^{n-1} \frac{[{}_hV(1+i) - {}_{h+1}V p_{x+h}] {}_h p_x}{(1+r)^h} \\
&= -E_0 + \frac{1}{1+r} \sum_{h=0}^{n-1} \frac{\{[G_h(1-c_h) - e_h](1+i) - (b_{h+1} + E_{h+1}) q_{x+h}\} {}_h p_x}{(1+r)^h} \\
&\quad - {}_0V + (1+i) \sum_{h=0}^{n-1} \frac{{}_hV {}_h p_x}{(1+r)^h} - \sum_{h=0}^{n-1} \frac{{}_{h+1}V {}_{h+1} p_x}{(1+r)^h} \\
&= -E_0 + \frac{1}{1+r} \sum_{h=0}^{n-1} \frac{\{[G_h(1-c_h) - e_h](1+i) - (b_{h+1} + E_{h+1}) q_{x+h}\} {}_h p_x}{(1+r)^h} \\
&\quad - {}_0V + (1+i) \sum_{h=0}^{n-1} \frac{{}_hV {}_h p_x}{(1+r)^h} - (1+r) \sum_{h=1}^{n} \frac{{}_hV {}_h p_x}{(1+r)^h} \\
&= -E_0 + \frac{1}{1+r} \sum_{h=0}^{n-1} \frac{\{[G_h(1-c_h) - e_h](1+i) - (b_{h+1} + E_{h+1}) q_{x+h}\} {}_h p_x}{(1+r)^h} \\
&\quad + (i-r) \sum_{h=0}^{n-1} \frac{{}_hV {}_h p_x}{(1+r)^{h+1}}
\end{aligned}$$

(where the last step follows from the fact that ${}_nV = 0$ for a term insurance).

Normally, $i < r$, and hence the coefficient $i - r$ in the last term is less than 0. Holding policy values lead to a penalty in NPV. The greater ${}_kV$ is, the greater the penalty is.

1.4 Using Profit Test to Compute Premiums and Reserves

We can use profit measures to determine premiums. For example, we may set the target IRR for a policy to be at least 15% and find the lowest premium such that this is satisfied. We may also set the policy to break even after 6 years and use a discounted payback period of 6 years to find the premium. The following example illustrates various premium principles that you have learnt so far.

Example 1.7. An insurer issues a 5-year fully discrete endowment insurance of \$10,000 to (60). The insurer assumes that initial expenses will be \$30, and renewal expenses, which are incurred at the beginning of the second and subsequent years in which a premium is payable, will be 2.5% of the premium. There is no settlement expense. The insurer holds net premium policy values, using an interest rate of 6%. The mortality rates used to calculate the premium and policy values are

$$q_{60} = 0.0138,\ q_{61} = 0.0150,\ q_{62} = 0.0164,\ q_{63} = 0.0179,\ \text{ and } q_{64} = 0.0195.$$

The funds invested for the policy are expected to earn interest rate of 7%.

You are given:

$$\text{Under } i = 7\%,\ \ddot{a}_{60:\overline{5}|} = 4.2659799 \text{ and } {}^2\ddot{a}_{60:\overline{5}|} = 3.7852826$$
$$\text{Under } i = 6\%,\ \ddot{a}_{60:\overline{5}|} = 4.34044137$$

(a) Suppose the insurer charges gross premium. Calculate the gross premium for the policy using an interest rate of 7%.

(b) Suppose the insurer charges percentile premium. Calculate the premium such that when a portfolio of such contracts is sold to 100 independent lifetimes aged (60), the probability that the aggregate prospective loss (calculated using $i = 7\%$) is greater than 0 is only 30%. Use a normal approximation. [Note: $\Phi^{-1}(0.7) = 0.5244$]

(c) Calculate the net premium and all net premium policy values, using an interest rate of 6%.

(d) Suppose that the insurer sets the level premium so that the profit margin on the policy is 15%, using a risk-adjusted discount rate of 10%. Calculate the premium under the following profit test basis:

Survival model: Same set of mortality rates as before.

Expenses:

Initial expenses of \$30 incorporated in the profit at time 0, no expenses for the profit during the first policy year, a renewal expense of 2.5% of the premium payable at the beginning of the year for the remaining policy years.

Interest on assets: 7%

In the calculation of profits, use net premium policy values obtained in (c).

Solution:

(a) Let G be the gross premium. By the equivalence principle,

$$10000A_{60:\overline{5}|} + 30 + 0.025G(\ddot{a}_{60:\overline{5}|} - 1) = G\ddot{a}_{60:\overline{5}|}.$$

where $A_{60:\overline{5}|} = 1 - \frac{0.07}{1.07} \times 4.2659799 = 0.7209172$. As a result,

$$G = \frac{10000A_{60:\overline{5}|} + 30}{0.975\ddot{a}_{60:\overline{5}|} + 0.025} = 1730.067.$$

(b) The prospective loss random variable for a single policy is

$$\begin{aligned} {}_0L &= 10000v^{(K+1)\wedge 5} + 30 + 0.025P(\ddot{a}_{\overline{(K+1)\wedge 5}|} - 1) - P\ddot{a}_{\overline{(K+1)\wedge 5}|} \\ &= \left(10000 + \frac{0.975P}{d}\right)v^{(K+1)\wedge 5} + 30 - \frac{0.975P}{d} - 0.025P. \end{aligned}$$

So,

$$\mathrm{E}({}_0L) = \left(10000 + \frac{0.975P}{d}\right)A_{60:\overline{5}|} + 30 - \frac{0.975P}{d} - 0.02P = 7239.172 - 4.18433044P,$$

and the variance is

$$\begin{aligned} \mathrm{Var}({}_0L) &= \left(10000 + \frac{0.975P}{d}\right)^2 \left({}^2A_{60:\overline{5}|} - (A_{60:\overline{5}|})^2\right) \\ &= 0.001208211\left(10000 + \frac{0.975P}{d}\right)^2, \end{aligned}$$

giving a standard deviation of $0.03475933\left(10000 + \frac{0.975P}{d}\right)$.

By normal approximation, ${}_0L^* \sim \mathrm{N}(100\mathrm{E}({}_0L), 100\mathrm{Var}({}_0L))$.

Since $\Pr({}_0L^* > 0) = 0.3$, the 70^{th} percentile of ${}_0L^*$ is 0. By normal approximation, the 70^{th} percentile is $100\mathrm{E}({}_0L) + 0.5244 \times 10\mathrm{SD}({}_0L)$. Therefore,

$$100(7239.172 - 4.18433044P) + 0.18227795(10000 + 14.9035714P) = 0,$$

and hence $P = 1745.76$.

(c) The net premium is $10000P_{60:\overline{5}|} = 10000\left(\frac{1}{\ddot{a}_{60:\overline{5}|}} - d\right) = 1737.87542$.

Starting with ${}_0V = 0$, Fackler's accumulation formula gives

$${}_1V = \frac{1737.87542 \times 1.06 - 138}{1 - 0.0138} = 1727.9943,$$

$${}_2V = \frac{(1727.9943 + 1737.87542) \times 1.06 - 150}{1 - 0.015} = 3577.4841,$$

$${}_3V = \frac{(3577.4841 + 1737.87542) \times 1.06 - 164}{1 - 0.0164} = 5561.4896,$$

$${}_4V = \frac{(5561.4896 + 1737.87542) \times 1.06 - 179}{1 - 0.0179} = 7696.0868.$$

Of course ${}_5V = 10000$.

(d) Let the annual premium be G. A simple calculation gives

k	1.1^{-k}	${}_kp_{60}$	$1.1^{-(k+1)}{}_kp_{60}$	$1.1^{-k}{}_kp_{60}$
0	1	1	0.909091	1
1	0.90909091	0.9862	0.815041	0.896546
2	0.82644628	0.971407	0.729832	0.802816
3	0.75131480	0.955475925	0.652603	0.717863
4	0.68301346	0.938372906	0.582656	0.640921

The APV of the premium, discounted using 10% interest, is

$$(1 + 0.896546 + 0.802816 + 0.717863 + 0.640921)G = 4.058146G.$$

Then we compute the profit vector for the contract:

$\mathrm{Pr}_0 = -30$,

$\mathrm{Pr}_1 = 1.07G - 138 - 0.9862 \times 1727.9943 = 1.07G - 1842.1480$

$$\begin{aligned}\mathrm{Pr}_2 &= 1.07(1727.9943 + 0.975G) - 150 - 0.985 \times 3577.4841\\ &= 1.04325G - 1824.868\end{aligned}$$

$$\begin{aligned}\mathrm{Pr}_3 &= 1.07(3577.4810 + 0.975G) - 164 - 0.9836 \times 5561.4896\\ &= 1.04325G - 1806.3765\end{aligned}$$

$$\begin{aligned}\mathrm{Pr}_4 &= 1.07(5561.4896 + 0.975G) - 179 - 0.9821 \times 7696.0868\\ &= 1.04325G - 1786.5330\end{aligned}$$

$\mathrm{Pr}_5 = 1.07(7696.0868 + 0.975G) - 10000 = 1.04325G - 1765.1871$

The NPV of the project is

$$\begin{aligned}&-30 + 0.909091\mathrm{Pr}_1 + 0.815041\mathrm{Pr}_2 + 0.729832\mathrm{Pr}_3 + 0.625203\mathrm{Pr}_4 + 0.582656\mathrm{Pr}_5\\ &= 3.8731G - 6704.768.\end{aligned}$$

The profit margin is $\dfrac{3.8731G - 6704.768}{4.058146G} = 0.15$. On solving, we get $G = 2053.92$.

□

Now we discuss how profit test can be used to compute reserves. As a preliminary, let us go back to Example 1.3.

Example 1.8. Consider the insurance policy in Example 1.3. The premium charged is 1034.7523. The profit test basis is changed to

(i) Mortality: $q_{40} = 0.08$, $q_{41} = 0.09$, $q_{42} = 0.10$, $q_{43} = 0.11$

(ii) Expenses: same as those in the premium basis

(iii) Interest: 6% annual effective.

(a) Determine the NPV for the policy, assuming that the company holds no policy value. Use a risk discount rate of 12%.

(b) Assume that the company holds net premium policy value determined from the following basis:

Mortality: $q_{40} = 0.09$, $q_{41} = 0.10$, $q_{42} = 0.12$, $q_{43} = 0.12$.

Interest: 3% annual effective.

Determine the NPV for the policy. Again use a risk discount rate of 12%.

Solution:

(a) The expected profits are

$$\begin{aligned}
\mathrm{Pr}_1 &= 0.95 \times 1034.7523 \times 1.06 - 10000 \times 0.08 = 241.9956, \\
\mathrm{Pr}_2 &= 0.95 \times 1034.7523 \times 1.06 - 10000 \times 0.09 = 141.9956, \\
\mathrm{Pr}_3 &= 0.95 \times 1034.7523 \times 1.06 - 10000 \times 0.10 = 41.9956, \\
\mathrm{Pr}_4 &= 0.95 \times 1034.7523 \times 1.06 - 10000 \times 0.11 = -58.0044,
\end{aligned}$$

We also have, under the profit test basis, $p_{40} = 0.92$, ${}_2p_{40} = 0.8372$, ${}_3p_{40} = 0.75348$, and the profit signature is $(-298.688, 241.9956, 130.6360, 35.1587, -43.7052)'$.

Under a risk discount rate of 12%, the NPV is

$$-298.688 + \frac{241.9956}{1.12} + \frac{130.636}{1.12^2} + \frac{35.1587}{1.12^3} - \frac{43.7052}{1.12^4} = 18.77.$$

(b) With the help of the following table,

k	q_{40+k}	${}_kp_{40}$	${}_{k\vert}q_{40} = {}_kp_{40}q_{40+k}$
0	0.09	1	0.09
1	0.10	0.91	0.091
2	0.12	0.819	0.09828
3	0.12	0.72072	0.0864864

it is easy to find that

$$\ddot{a}_{40:\overline{4}|} = \sum_{k=0}^{3} \frac{{}_kp_{40}}{1.03^k} = 3.31504209, \; A^{1}_{40:\overline{4}|} = \sum_{k=0}^{3} \frac{{}_{k|}q_{40}}{1.03^{k+1}} = 0.33993704.$$

This gives a net premium of 1025.43807, and the net premium policy values are

$$_0V = 0,\ _1V = \frac{1025.43807 \times 1.03 - 10000 \times 0.09}{0.91} = 171.6497,$$

and similarly, $_2V = 258.8893$, $_3V = 139.6105$, $_4V = 0$. The profits at time 1, 2, 3 and 4 become:

$$\begin{aligned}
\mathrm{Pr}_1 &= 241.9956 + 1.06 {}_0V - 0.92 {}_1V = 84.0779 \\
\mathrm{Pr}_2 &= 141.9956 + 1.06 {}_1V - 0.91 {}_2V = 88.3550 \\
\mathrm{Pr}_3 &= 41.9956 + 1.06 {}_2V - 0.9 {}_3V = 190.7688 \\
\mathrm{Pr}_4 &= -58.0044 + 1.06 {}_3V - 0.89 {}_4V = 89.9827
\end{aligned}$$

The profit signature is

$$\mathbf{\Pi} = (-298.688,\ 84.0779,\ 81.2866,\ 159.7117,\ 67.8001)'.$$

Under a discount rate of 12%, the NPV is

$$-298.688 + \frac{84.0779}{1.12} + \frac{81.2866}{1.12^2} + \frac{159.7117}{1.12^3} + \frac{67.8001}{1.12^4} = -2.05.$$

□

Holding reserves reduces profitability because reserves earn a lower rate of return. Insurance companies would sometimes like to hold the lowest reserve possible, as long as the emerging profits for all years (except Pr_0) are non-negative. The process for determining policy values using this principle is called **zeroization** and the resulting policy values are called **zeroized reserves**.

Let us use the previous example as an illustration. Suppose again that the premium is 1034.7523 and the profit test basis is as specified in the example. (We do not need the basis specified in (b) because we now determine reserve values using another principle.)

If the company holds no reserves, then $\mathrm{Pr}_4 < 0$. Let $_tV^Z$ be the reserve needed to zeroize Pr_4.

$$\mathrm{Pr}_4 = -58.0044 + 1.06 {}_3V^Z = 0,$$

giving $_3V^Z = 54.72113$. Then we move to Pr_3:

$$\begin{aligned}
\mathrm{Pr}_3 = 41.9956 + 1.06 {}_2V^Z - 0.9 {}_3V^Z \geq 0 \\
{}_2V^Z \geq (0.9 \times 54.72113 - 41.99566)/1.06 = 6.842846.
\end{aligned}$$

So we set $_2V^Z = 6.842846$. Then we move to Pr_2:

$$\begin{aligned}
\mathrm{Pr}_2 = 141.9956 + 1.06 {}_1V^Z - 0.91 {}_2V^Z \geq 0 \\
{}_1V^Z \geq (0.91 \times 6.842846 - 141.99566)/1.06 = -128.08
\end{aligned}$$

So we can set $_1V^Z = 0$. Finally we move to Pr_1:

$$\begin{aligned}
\mathrm{Pr}_1 = 241.9956 + 1.06 {}_0V^Z - 0.92 {}_1V^Z \geq 0 \\
{}_0V^Z \geq -241.9956
\end{aligned}$$

So we can set $_1V^Z = 0$.

Now we can redo the profit test using this set of reserves:

Obviously, $\mathrm{Pr}_3 = \mathrm{Pr}_4 = 0$. The profits at time 1 and 2 become:

$$\mathrm{Pr}_1 = 241.9956 + 1.06{}_0V - 0.92{}_1V = 241.9956$$
$$\mathrm{Pr}_2 = 141.9956 + 1.06{}_1V - 0.91{}_2V = 135.7686$$

The profit signature is

$$\mathbf{\Pi} = (-298.688,\ 241.9956,\ 124.9071,\ 0,\ 0)'.$$

Under a discount rate of 12%, the NPV is

$$-298.688 + \frac{241.9956}{1.12} + \frac{124.9071}{1.12^2} = 16.95,$$

which is in between 18.77 and -2.05.

Zeroization of Profit Vector

Suppose that the profit vector calculated without allowances for reserves is

$$\mathbf{Pr} = (\mathrm{Pr}_0,\ \mathrm{Pr}_1,\ \mathrm{Pr}_2,\ \mathrm{Pr}_3,\ \ldots)'.$$

Then for term policy,

$${}_tV^Z = \max\left(\frac{p_{x+t\,t+1}V^Z - \mathrm{Pr}_{t+1}}{1+i_t}, 0\right).$$

For endowment policy, use the above for $t < n-1$. For $t = n$,

$${}_{n-1}V^Z = \max\left(-\frac{\mathrm{Pr}_n}{1+i_{n-1}}, 0\right).$$

The special formula for $t = n-1$ for endowment policy is owing to the form of Pr_n preceding Example 1.3.

1.5 Exercise 1

1. Consider the set up in Example 1.2. Decompose the gain in the second policy year into profit / loss attribute to interest expense and mortality in the following order:

   Expense, mortality, and finally interest.

2. (MLC Sample #300) For a fully discrete 20-year term life insurance of 10,000 on (40), you are given:

   (i) The gross premium is 87.

   (ii) Values in year 4:

	Anticipated	Actual
Expenses as a percent of premium	3%	2.5%
q_{43}	0.0035	0.0025
Annual effective rate of interest	5%	4%

   (iii) Reserves, which are gross premium reserves, are

End of year	Reserve
3	83.30
4	141.57

   A company issued the 20-year term life insurance to 850 lives age 40 with independent future lifetimes. At the end of the 3^{rd} year 800 insurances remain in force. Calculate the total gain from mortality, interest and expenses in year 4 from the 800 insurances.

   (A) 6,580 (B) 6,910 (C) 6,970 (D) 7,030 (E) 7,090

3. Refer to the set up in the previous question. Decompose the gain in the following orders:

   (a) interest, then expenses, and finally mortality;

   (b) mortality, expenses, and finally interest;

   (c) mortality, interest, and finally expenses.

4. (MLC Sample #303 – #305) (65) purchases a whole life annuity that pays 1000 at the end of each year. You are given:

(i) The gross single premium is 15,000.

(ii) 1000 such policies are in force at the beginning of year 10.

(iii)

	Anticipated experience	Actual experience
Mortality	$q_{74} = 0.01$	12 deaths
Interest	$i = 0.06$	$i = 0.05$
Expenses	50 per policy	60 per policy

(iv) Expenses are paid at the end of each year for any policyholder who does not die during the year.

(v) Reserves are gross premium reserves.

(vi) The reserve at the end of the ninth year is 10,994.49.

You calculate the gain from interest during year 10, with the gain from interest calculated prior to the calculation of gain from any other sources.
Calculate the gain from interest.

(A) −112,000 (B) −111,000 (C) −110,000 (D) −109,000 (E) −120,280

5. Refer to Question 4. You calculate the gain from expenses during year 10, assuming the gains from interest has already been calculated and the gain from mortality is yet to be calculated.
Calculate the gain from expenses.

(A) −9,910 (B) −9,900 (C) −9,890 (D) −9,880 (E) −9,870

6. Refer to Question 4. You calculate the gain from mortality during year 10, assuming the gains from interest and expense have already been calculated.
Calculate the gain from mortality.

(A) 19,540 (B) 21,540 (C) 21,560 (D) 23,540 (E) 23,560

7. Consider the profit signature in Example 1.3 under the case with reserves:

$$\Pi = (-298.688, 103.335, 96.357, 79.035, 71.132)'.$$

Calculate the following profit measures under a risk discount rate of 8% effective:

(a) the expected present value of the profits at issue

(b) internal rate of return

(c) discount payback period

(d) profit margin

8. Consider a 6-year annual level premium endowment insurance of 10000 on (50). The premium basis is as follows:

(i) Mortality: $q_{49} = 0.005$, $q_{50} = 0.006$, $q_{51} = 0.006$, $q_{52} = 0.007$,
$q_{53} = 0.008$, $q_{54} = 0.008$, $q_{55} = 0.009$.

(ii) Expenses: Pre-contract expenses: 50 plus 25% of the first premium
Renewal and other expenses: 5% of the premiums, including the first at the beginning of each period. Settlement expenses of 10 is payable at the end of the year whenever a benefit is made.

(iii) Interest: 5% annual effective.

You are also given the following table:

k	q_{50+k}	${}_kp_{50}$	${}_{k\|}q_{50} = {}_kp_{50}q_{50+k}$	1.05^{-k}
0	0.006	1	0.006	1
1	0.006	0.994	0.005964	0.952381
2	0.007	0.988036	0.006916	0.907029
3	0.008	0.981120	0.007849	0.863838
4	0.008	0.973271	0.007786	0.822702
5	0.009	0.965485	0.008689	0.783526
6		0.956795		0.746215

(a) Calculate the gross premium.

(b) Calculate the profits at time 0, 1, 2, 3, 4, 5 and 6, assuming that the insurance company holds no reserves for the policy and that the expenses assumption under the profit test basis is the same as the premium basis. For the mortality assumptions, use ${}^Pq_x = q_{x-1}$, where ${}^Pq_x$ is the probability of death at age x under the profit test basis. For interest, use 10% for insurer's assets.

(c) Calculate the FPT reserves under an interest of 4% and the mortality assumption as in the premium basis.

(d) Calculate profit vector for the policy, assuming that the insurance company holds FPT reserves computed in (c). Use the same profit test basis as in (b).

(e) Find the profit signature for the policy.

(f) Calculate the following profit measures, assuming that the risk discount rate is 12% effective per year:

(i) the expected present value of the profits at issue

(ii) profit margin

9. Consider the set up in Example 1.5. Calculate the following profit measures:
   (a) the net present value under an annual effective risk discount rate of 10%,
   (b) the profit margin under an annual effective risk discount rate of 10%,
   (c) the IRR for the policy.

10. For a fully discrete 2-year life insurance policy on (40) you are given:
   (i) All cash flows are annual.
   (ii) The annual gross premium is 100.
   (iii) Profits and premiums are discounted at an annual effective rate of r.
   (iv) The profit vector:

Time in years	0	1	2
Profit	-275	80	238

   (v) $p_{40} = 0.99722$ under the profit test basis.
   (vi) The profit margin is -8.66%.

   Calculate r.

11. For a fully discrete 4-year life insurance policy on (50) you are given:
   (i) All cash flows are annual.
   (ii) The annual gross premium is 100 for the first two years and 120 thereafter.
   (iii) Profits and premiums are discounted at an annual effective rate of r.
   (iv) The profit signature:

Time in years	0	1	2	3	4
Profit signature	-275	-76	122	250	218

   (v) Under the profit test basis, $q_{50} = 0.0060$, $q_{51} = 0.0065$ and $q_{52} = 0.0070$.
   (vi) The profit margin is 24.8%.

   Calculate r.

12. (MLC Sample #289) For a 3-year term insurance of 1,000,000 on (60), you are given:

(i) The death benefit is payable at the end of the year of death.

(ii) $q_{60+t} = 0.014 + 0.001t$

(iii) Cash flows are accumulated at annual effective rate of interest of 0.06.

(iv) The annual gross premium is 14,500.

(v) Pre-contract expenses are 1000 and are paid at time 0.

(vi) Expenses after issue are 100 payable immediately after the receipt of each gross premium.

(vii) The reserve is 700 at the end of the first and second years.

(viii) Profits are discounted at annual effective rate of interest of 0.10.

Calculate the net present value of the policy.

(A) -155 (B) -174 (C) -177 (D) -187 (E) -216

13. (MLC Sample #292) For a fully discrete 3-year term life insurance policy on (40) you are given:

(i) All cash flows are annual.

(ii) The annual gross premium is 1000.

(iii) Profits and premiums are discounted at an annual effective rate of 0.12.

(iv) The profit vector: $\text{Pr} = (-400, 150, 274, 395)'$

(v) The profit signature: $\mathbf{\Pi} = (-400, 150, 245, 300)'$

Calculate the profit margin.

(A) 4.9% (B) 5.3% (C) 5.9% (D) 6.6% (E) 9.7%

14. An insurer issues a 2-year term insurance of \$100,000 to a life aged 60. Premium of \$185 is payable every six months, while the death benefit is payable at the moment of death. Under a reserve basis,

$$_{0.5}V = 14.4,\ _1V = 19.7,\ _{1.5}V = 15.2.$$

To conduct a profit test, all deaths are assumed to occur in the middle of each half year period. Under the profit test basis, the insurer assumes an initial expense of 10% of the premium, and 5% of the subsequent premiums. The funds invested for the policy are expected to earn interest rate of 7% per year. The survival model used is

$$\mu_x = 0.0002 + b \times 1.125^x.$$

Let Pr_t be the emerging profit at time t for a policy that is still in force at time $t - 0.5$, and Π_t be the expected profit over $(t - 0.5,\ t)$, at issue of the policy. You are given that $\text{Pr}_2 = 4.307441$ and $\Pi_{1.5} = 4.235106$.

(a) Calculate the profit signature.

(b) Calculate the internal rate of return for this policy.

15. Consider a 3-year fully discrete endowment insurance of 10,000 on (60). The insurer holds net premium reserves, using an interest rate of 6%. The survival model used to calculate the premium and net premium reserves is

$$q_{60} = 0.0138,\ q_{61} = 0.0150,\ q_{62} = 0.0164$$

The funds invested for the policy are expected to earn interest rate of 8%.

You are given: Under $i = 8\%, \ddot{a}_{60:\overline{3}|} = 2.7459731$ and ${}^2\ddot{a}_{60:\overline{3}|} = 2.55952069$

Under $i = 6\%, \ddot{a}_{60:\overline{3}|} = 2.7949261$

(a) Suppose the insurer charges gross premium. Calculate the gross premium for the policy using an interest rate of 8%, and assuming that initial expenses will be $30, and renewal expenses, which are incurred at the beginning of the second and third year in which a premium is payable, will be 3% of the premium, and that there is no settlement expense.

(b) Suppose the insurer charges percentile premium. Calculate the premium such that when a portfolio of 500 such contracts is sold to 500 independent lifetimes aged 60, the probability that the aggregate prospective loss (calculated using $i = 8\%$) is greater than 100,000 is only 20%. Use the same expense assumptions as in part (a). Use a normal approximation.

[Note: $\Phi^{-1}(0.8) = 0.84162$]

(c) Calculate the net premium and all net premium reserves, using an interest rate of 6%.

(d) Suppose that the insurer sets the level premium so that the profit margin on the policy is 15%, using a risk-adjusted discount rate of 20%. Calculate the premium under the following profit test basis:

Survival model: same as that in the reserve basis.

Expenses: initial expenses of 30 incorporated in the profit at time 0, no initial expenses for the profit during the first policy year, a renewal expense of 3% of the premium payable at the beginning of the year for the 2nd and 3rd year, and a settlement expense of 100 if death occurs during the 1st and the 2nd policy year.

Interest on assets: 8%

In the calculation of profits, use net premium reserves obtained in part (c).

16. A 5-year term insurance with annual cash flows issued to a life aged x produces the profit vector.

$$\mathbf{Pr} = (-329,\ 450,\ 229,\ 94, -55, -217)',$$

This profit vector has been calculated without allowances for reserves. Under the profit test basis, the mortality rates are given by the formula

$$q_{x+t} = 0.013 + 0.001t,\ \ t \geq 0.$$

and the interest rate is 5% per year. The insurer sets reserves by zeroization.

Calculate the zeroized reserves and the revised profit vector after allowance for reserves.

17. Consider a 5-year non-level premium term insurance of 10000 issued to a life aged 60. Under the premium basis, the premiums for the first, second and third policy year are 64 each, while the premiums for the fourth and the fifth policy year are 128 each.

(a) Consider the following profit test basis

(a) Mortality: $q_{60} = 0.005$, $q_{61} = 0.007$, $q_{62} = 0.008$, $q_{63} = 0.009$, $q_{64} = 0.01$

(b) Expenses: Pre-contract expenses: 30% of the first premium Renewal and other expenses: 5% of the premiums, excluding the first, at the beginning of each period. Settlement expenses of 10 is payable at the end of the year whenever a benefit is made.

(c) Interest: 4% during the first two years, 5% during the next three years.

Calculate the expected profit at the end of each year given that the insurance is in force at age 60. Assume that the insurer set no reserves for each policy in force.

(b) Calculate the NPV for the policy using a risk discount rate of 10%.

(c) Calculate the zeroized reserves and the revised profit signature after allowance for reserves.

(d) Calculate the NPV for the policy using the revised profit signature. Use a risk discount rate of 10%. What is the IRR for the policy after allowance for reserves?

18. (MLC Sample Structural Question #16) An insurer issues a fully discrete four year term insurance contract to a life aged 50. The face amount is 100,000 and the gross annual premium for the contract is 660 per year. The company uses the following assumptions to analyze the emerging surplus of the contract.

Interest:	7% per year
Initial expenses:	80 immediately before first premium
Renewal expenses:	14 incurred on each premium date, including the first
Mortality:	Illustrative Life Table (you can find this in the appendix)
Lapses:	None

(a) Assume first that the insurer sets level reserves of ${}_tV^L = 50, t = 0, 1, 2, 3$ for each policy in force.

(i) Calculate the profit vector for the contract.

(ii) Calculate the profit signature for the contract.

(iii) Calculate the NPV assuming a risk discount rate of 10% per year.

(b) The insurer is considering a different reserve method. The reserves would be set by zeroizing the emerging profits under the profit test assumptions, subject to a minimum reserve of 0. Calculate ${}_tV^Z$ for $t = 3, 2, 1, 0$.

(c) State with reasons whether the NPV in (a)(iii) would increase or decrease using the zeroized reserves from part (b), but keeping all other assumptions as in part (a).

[8+4+2=12 points]

1.6 Solutions to Exercise 1

1. Expense: $-475 \times 0.5 \times 1.06 + 475 \times 0.09 \times 1 = -209$
   Mortality: $475 \times (1004 - 470.5737) \times (0.09 - 0.10105263) = -2800.4877$
   Interest: $475[182.2313 + 350(1 - 0.05) - 2.5] \times (0.05 - 0.06) = -2433.0987$

2. At time 3, the insurance company has 800×83.3.
   The amount of premium received is $87(1 - 2.5\%) \times 800$.
   So the amount of asset is 134,500.
   At the end of the year, the amount grows to $134{,}500 \times 1.04 = 139{,}880$.
   Death claims amounts to $800 \times 0.0025 \times 10000 = 20{,}000$.
   Reserve required at time 4 is $800 \times (1 - 0.0025) \times 141.57 = 112{,}972.86$.
   So the actual profit is $139{,}880 - (20{,}000 + 112{,}972.86) = 6907.14$.
   The expected profit is

   $$800[83.3 + 87(1 - 0.03)] \times 1.05 - 800 \times (0.0035 \times 10000 + 0.9965 \times 141.57) = -0.004$$

   which can be treated as zero. So the gain is 6907.14, and the correct answer is (B).

3. (a) Interest component: $800 \times [83.3 + 87(1 - 0.03)] \times (-0.01) = -1{,}341.52$
   Expenses component: $800 \times 87 \times 0.5\% \times 1.04 = 361.92$
   Mortality component: $800 \times (10000 - 141.57) \times (0.001) = 7{,}886.744$

   (b) Mortality component: $800 \times (10000 - 141.57) \times 0.001 = 7{,}886.744$
   Expenses component: $800 \times 87 \times 0.5\% \times 1.05 = 365.4$
   Interest component: $800 \times (83.3 + 87 \times 0.975) \times (-0.01) = -1{,}345$

   (c) Same as (a).

4. There is no cash flow at the beginning of the 10^{th} year because this is a single-premium policy. Interest is earned on the gross premium reserve. The interest gain is the

   $$\begin{aligned} 1000 \times (0.05 - 0.06) \times 10994.49 &= 1000 \times (-0.01) \times 10994.49 \\ &= -109{,}944.9. \end{aligned}$$

   Hence, the answer is (C).

5. We assume that mortality is as anticipated. Since the number of survivors anticipated at the end of the 10^{th} year is $1000 \times 0.99 = 990$, and expenses are paid only for survivors, the gain from expenses is $(50 - 60) \times 990 = -9900$. Hence, the answer is (B).

6. By the recursive relation for gross premium policy values,

$$
\begin{aligned}
{}_9V \times 1.06 &= 0.99({}_{10}V + 1000 + 50), \\
{}_{10}V &= 10{,}994.49 \times 1.06/0.99 - 1050 = 10{,}721.878.
\end{aligned}
$$

The gain from mortality, after accounting for interest and expense, is

$$-N(b_{h+1} + \hat{E}_{h+1} - {}_{h+1}V)(\hat{q}_{x+h} - q_{x+h}).$$

We have Nq_{x+h} = Anticipated number of deaths $= 1000 - 990 = 10$, and actual number of deaths $N\hat{q}_{x+h} = 12$. Note, however, that since it is an annuity policy, if death occurs, the insurance company needs not pay annuity payment, and also saves the expense. This means we should flip the sign of b and E, which amounts to

$$(-1000 - 60 - 10721.878) \times (10 - 12) = 23{,}563.756.$$

The answer is (E).

7. The profit signature is $(-298.688, 103.335, 96.357, 79.035, 71.132)'$.

(a) The expected present value of profits at issue plays the role of NPV for deterministic cash flows. The NPV at 8% is

$$-298.688 + \frac{103.335}{1.08} + \frac{96.357}{1.08^2} + \frac{79.035}{1.08^3} + \frac{71.132}{1.08^4} = -5.372$$

(b) By using a financial calculator, the IRR is found to be 7.13%. The IRR is unique because the elements in the profit signature change sign only once.

(c) Since the NPV is negative, the discounted payback period is infinite. Put it in another way, the project never pays back.

(d) The premium is 1034.7523. The APV of premiums is

$$1034.7523\left(1 + \frac{0.92}{1.08} + \frac{0.8372}{1.08^2} + \frac{0.75348}{1.08^3}\right) = 3277.839.$$

So the profit margin is $-5.372/3277.839 = -0.164\%$.

8. (a) $\ddot{a}_{50:\overline{6}|} = \sum_{k=0}^{5} 1.05^{-k}{}_kp_{50} = 5.247567$, $A_{50:\overline{6}|} = 1 - \frac{5}{105} \times 5.247567 = 0.750116$.

By the equivalence principle,

$$10010A_{40:\overline{6}|} + 50 + 0.25G = 0.95G\ddot{a}_{50:\overline{6}|}.$$

On solving, we get $G = 1596.27412$.

(b) $\mathrm{Pr}_0 = -50 - 0.25G = -449.069$

$\mathrm{Pr}_1 = 1.1 \times 0.95G - 10010 \times 0.005 = 1618.057$ (note: we use the mortality ${}^Pq_{50} = q_{49}$)

$\mathrm{Pr}_2 = 1.1 \times 0.95G - 10010 \times 0.006 = 1608.047$

$\mathrm{Pr}_3 = 1.1 \times 0.95G - 10010 \times 0.006 = 1608.047$

$\mathrm{Pr}_4 = 1.1 \times 0.95G - 10010 \times 0.007 = 1598.037$

$\mathrm{Pr}_5 = 1.1 \times 0.95G - 10010 \times 0.008 = 1588.026$

$\mathrm{Pr}_6 = 1.1 \times 0.95G - 10010 = -8341.893$

(c) For FPT reserve, ${}_0V^{FPT} = 0$, ${}_1V^{FPT} = 0$, and ${}_kV$ = the time-k net premium reserve for a 5-year endowment policy of 10000 with issued to a life aged 51.

k	q_{51+k}	${}_kp_{51}$
0	0.006	1
1	0.007	0.994
2	0.008	0.987042
3	0.008	0.979146
4	0.009	0.971312

$$\ddot{a}_{51:\overline{5}|} = \sum_{k=0}^{4} 1.04^{-k}\,{}_kp_{51} = 4.569084,\ P_{51:\overline{5}|} = \frac{1}{4.569084} - \frac{0.04}{1.04} = 0.1804007$$

As a result, by Fackler's accumulation formula,

$$\begin{aligned}
{}_2V^{FPT} &= 10000\,{}_1V_{51:\overline{5}|} = 10000\frac{0.1804007 \times 1.04 - 0.006}{1 - 0.006} = 1827.130\\
{}_3V^{FPT} &= 10000\,{}_2V_{51:\overline{5}|} = 10000\frac{(0.182713 + 0.1804007) \times 1.04 - 0.007}{1 - 0.007} = 3732.510\\
{}_4V^{FPT} &= 10000\,{}_3V_{51:\overline{5}|} = 10000\frac{(0.373251 + 0.1804007) \times 1.04 - 0.008}{1 - 0.008} = 5723.768\\
{}_5V^{FPT} &= 10000\,{}_4V_{51:\overline{5}|} = 10000\frac{(0.5723768 + 0.1804007) \times 1.04 - 0.008}{1 - 0.008} = 7811.377
\end{aligned}$$

And of course ${}_6V^{FPT} = 10000$.

(d) There is no change to Pr_0 and Pr_1 because the reserves at time 0 and time 1 are zero.

$\mathrm{Pr}_2 = 1608.047 - (1 - 0.006) \times 1827.13 = -208.120$

$\mathrm{Pr}_3 = 1608.047 + 1.1 \times 1827.13 - (1 - 0.006) \times 3732.51 = -92.225$

$\mathrm{Pr}_4 = 1598.037 + 1.1 \times 3732.51 - (1 - 0.007) \times 5723.768 = 20.096$

$\mathrm{Pr}_5 = 1588.026 + 1.1 \times 5723.768 - (1 - 0.008) \times 7811.377 = 135.285$

$\mathrm{Pr}_6 = -8341.893 + 1.1 \times 7811.377 = 250.622$

(Note that in Pr_6, there is no need to subtract the ${}_6p_x \times {}_6V$ term because the survival benefit has already been taken into account in the calculation in (b))

(e) We build the following table:

k	${}^{P}q_{50+k}$	${}^{P}{}_kp_{50}$	Pr_k	$\Pi_k = \mathrm{Pr}_k \times {}^{P}{}_{k-1}p_{50}$
0	0.005	1	−449.069	−449.069
1	0.006	0.995	1618.057	1618.057
2	0.006	0.98903	−208.120	−207.079
3	0.007	0.983096	−92.225	−91.213
4	0.008	0.976214	20.096	19.756
5	0.008	0.968404	135.285	132.067
6			250.622	242.703

(f) (i) The NPV is

$$-449.069+\frac{1618.057}{1.12}-\frac{207.079}{1.12^2}-\frac{91.213}{1.12^3}+\frac{19.756}{1.12^4}+\frac{132.067}{1.12^5}+\frac{242.703}{1.12^6}=976.00.$$

(ii) The APV of premiums is

$$1596.274\left(1+\frac{0.995}{1.12}+\frac{0.98903}{1.12^2}+\frac{0.983096}{1.12^3}+\frac{0.976214}{1.12^4}+\frac{0.968404}{1.12^5}\right)=7257.444.$$

So the profit margin is $976/7257.445 = 13.4\%$.

9. (a) The profit signature is $(-400,\ 140.913083,\ 120.863230,\ 121.202159,\ 121.541005)'$.

The NPV is

$$-400+\frac{140.913083}{1.1^{0.25}}+\frac{120.863230}{1.1^{0.5}}+\frac{121.202159}{1.1^{0.75}}+\frac{121.541005}{1.1}=76.166.$$

(b) The NPV of revenues is

$$200\left(1+\frac{0.999405833}{1.1^{0.25}}+\frac{0.998796394}{1.1^{0.5}}+\frac{0.998171264}{1.1^{0.75}}\right)=771.499756.$$

So, the profit margin is $76.166/771.499756 = 9.8725\%$.

(c) We let r be the IRR. The yield equation is

$$-400+\frac{140.913083}{(1+r)^{0.25}}+\frac{120.863230}{(1+r)^{0.5}}+\frac{121.202159}{(1+r)^{0.75}}+\frac{121.541005}{1+r}=0.$$

However, in this form we cannot use a financial calculator to solve for r. So, we get i be the interest rate per quarter. That is, $1+r=(1+i)^4$. Then

$$-400+\frac{140.913083}{1+i}+\frac{120.863230}{(1+i)^2}+\frac{121.202159}{(1+i)^3}+\frac{121.541005}{(1+i)^4}=0.$$

A financial calculator would give $i = 10.244649\%$ as the solution. The annual effective yield is $r=(1+i)^4-1=47.716\%$.

10. The NPV of the premiums is $100\left(1+\dfrac{0.99722}{1+r}\right)$.

The profit vector is $(-275,\ 80,\ 238)'$. The profit signature is

$$(-275,\ 80,\ 238\times 0.99722)'=(-275,\ 80,\ 237.33836)'.$$

So, the NPV is $-275+\frac{80}{1+r}+\frac{237.33836}{(1+r)^2}$.

Since the profit margin is -8.66%,

$$-275+\frac{80}{1+r}+\frac{237.33836}{(1+r)^2}=-0.0866\times 100\left(1+\frac{0.99722}{1+r}\right)$$

$$\frac{237.33836}{(1+r)^2}+\frac{88.635925}{1+r}-266.34=0$$

By using a financial calculator, r is found to be 12.494%.

[Alternatively, by using quadratic formula, $\frac{1}{1+r}=0.888939626$ or -1.26 (rejected). This would again give $r=0.12494$. However, if the policy has a term greater than 2 years, you cannot solve the equation directly but have to rely on a financial calculator.]

11. The profit signature is $\mathbf{\Pi} = (-275, -76, 122, 250, 218)'$.

The survival probabilities are

t	1	2	3
${}_tp_{50}$	0.994	0.987539	0.9806262

So, the APV of the premiums, under a risk discount rate of r, is

$$100\left(1+\frac{0.994}{1+r}+\frac{0.987539\times 1.2}{(1+r)^2}+\frac{0.9806262\times 1.2}{(1+r)^3}\right).$$

The NPV of the policy is $-275-\dfrac{76}{1+r}+\dfrac{122}{(1+r)^2}+\dfrac{250}{(1+r)^3}+\dfrac{218}{(1+r)^4}$.

Since the profit margin is 24.8%,

$$\begin{aligned}-275-\frac{76}{1+r}&+\frac{122}{(1+r)^2}+\frac{250}{(1+r)^3}+\frac{218}{(1+r)^4}\\ &=0.248\times 100\left(1+\frac{0.994}{1+r}+\frac{0.987539\times 1.2}{(1+r)^2}+\frac{0.9806262\times 1.2}{(1+r)^3}\right)\end{aligned}$$

or

$$299.8+\frac{100.6512}{1+r}-\frac{92.6108}{(1+r)^2}-\frac{220.817}{(1+r)^3}-\frac{218}{(1+r)^4}=0.$$

By using a financial calculator, r is found to be 10.0035%.

12. Statement (vi) in the question specifies the expenses of 100, including the one incurred at time 0^+, are not acquisition expenses. Hence they are incorporated in the calculation of Pr_1, Pr_2, etc.

$\text{Pr}_0 = -1000$

$\text{Pr}_1 = (14500-100)\times 1.06-(0.014\times 1000000+0.986\times 700) = 573.8$

$\text{Pr}_2 = (14500+700-100)\times 1.06-(0.015\times 1000000+0.985\times 700) = 316.5$

$\text{Pr}_3 = (14500+700-100)\times 1.06-(0.016\times 1000000+0.984\times 0) = 6$

The NPV, computed under a risk discount rate of 10%, is

$$-1000+\frac{573.8}{1.1}+\frac{316.5}{1.1^2}\times 0.986+\frac{6}{1.1^3}\times 0.986\times 0.985=-216.$$

13. By comparing (iv) and (v), we get

$$p_{40}=245/274 \text{ and } {}_2p_{40}=300/395.$$

The NPV is $-400+\dfrac{150}{1.12}+\dfrac{245}{1.12^2}+\dfrac{300}{1.12^3}=142.775$.

The APV of premiums is $1000\left(1+\dfrac{245/274}{1.12}+\dfrac{300/395}{1.12^2}\right)=2403.821$.

So, the profit margin is $142.775/2403.821 = 5.94\%$.

14. (a) Since

$\text{Pr}_2 = (15.2 + 185 \times 0.95) \times 1.07^{0.5} - {}_{0.5}p_{61.5} \times 0 - {}_{0.5}q_{61.5} \times 100000 \times 1.07^{0.25} = 4.307441,$

on solving, we get ${}_{0.5}q_{61.5} = 0.001899721$. Then we use ${}_tp_x = \exp\left[-At - \frac{B}{\ln c}c^x(c^t - 1)\right]$ to obtain

$$\ln 0.998100278 = -0.0001 - \frac{b \times 1.125^{61.5}}{\ln 1.125}(1.125^{0.5} - 1).$$

This gives $b = 2.5 \times 10^{-6}$.

Now we can find the remaining probabilities:

${}_{0.5}p_{60} = 0.99839152$, ${}_{0.5}p_{60.5} = 0.99830009$, ${}_{0.5}p_{61} = 0.99820312$,

$\text{Pr}_0 = -18.5$

$$\begin{aligned}\text{Pr}_{0.5} &= 185 \times 1.07^{0.5} - 0.99839152 \times 14.4 - 0.00160848 \times 100000 \times 1.07^{0.25} \\ &= 13.396820\end{aligned}$$

$$\begin{aligned}\text{Pr}_1 &= (14.4 + 185 \times 0.95) \times 1.07^{0.5} - 0.99830009 \times 19.7 \\ &\quad - 0.00169991 \times 100000 \times 1.07^{0.25} \\ &= 4.135382\end{aligned}$$

The profit signature is

$$\begin{aligned}\mathbf{\Pi} &= (-18.5,\ 13.396820,\ {}_{0.5}p_{60} \times 4.135382,\ 4.235016,\ {}_{1.5}p_{60} \times 4.307441)' \\ &= (-18.5,\ 13.396820,\ 4.128730,\ 4.235016,\ 4.285488)'\end{aligned}$$

(b) Using the IRR function of a financial calculator, we solve the equation

$$-18.5 + \frac{13.396820}{1+i} + \frac{4.128730}{(1+i)^2} + \frac{4.235016}{(1+i)^3} + \frac{4.285488}{(1+i)^4} = 0.$$

The calculator reports an IRR of 20.1801741%.

This gives an annual effective rate of $1.201801741^2 - 1 = 44.432\%$.

15. (a) Let G be the gross premium. By the equivalence principle,

$$10000A_{60:\overline{3}|} + 30 + 0.03G(\ddot{a}_{60:\overline{3}|} - 1) = G\ddot{a}_{60:\overline{3}|},$$

where $A_{60:\overline{3}|} = 1 - \dfrac{0.08}{1.08} \times 2.7459731 = 0.79659459$. As a result,

$$G = \frac{10000A_{60:\overline{3}|} + 30}{0.97\ddot{a}_{60:\overline{3}|} + 0.03} = 2968.5046.$$

(b) ${}^2A_{60:\overline{3}|} = 1 - {}^2d^2\ddot{a}_{60:\overline{3}|} = 1 - (1 - {}^2v)^2\ddot{a}_{60:\overline{3}|} = 1 - (1 - 1.08^{-2}) \times 2.55952069 = 0.63485576$

The prospective loss random variable for a single policy is

$$\begin{aligned}{}_0L &= 10000v^{(K+1)\wedge 3} + 30 + 0.03P(\ddot{a}_{\overline{(K+1)\wedge 3}|} - 1) - P\ddot{a}_{\overline{(K+1)\wedge 3}|} \\ &= \left(10000 + \frac{0.97P}{d}\right)v^{(K+1)\wedge 3} + 30 - \frac{0.97P}{d} - 0.03P\end{aligned}$$

So,

$$\mathrm{E}({}_0L) = \left(10000 + \frac{0.97P}{d}\right) A_{60:\overline{3}|} + 30 - \frac{0.97P}{d} - 0.03P = 7995.9459 - 2.6935939P,$$

and the variance is

$$\mathrm{Var}({}_0L) = \left(10000 + \frac{0.97P}{d}\right)^2 ({}^2A_{60:\overline{3}|} - (A_{60:\overline{3}|})^2) = 0.000292823 \left(10000 + \frac{0.97P}{d}\right)^2,$$

giving a standard deviation of $0.0171121 \left(10000 + \frac{0.97P}{d}\right)$.

By normal approximation, ${}_0L^* \sim \mathrm{N}(500\mathrm{E}({}_0L), 500\mathrm{Var}({}_0L))$. We know that 100000 is the 80$^{\text{th}}$ percentile of ${}_0L^*$. On the other hand, the percentile is

$$500\mathrm{E}({}_0L) + 0.84162\sqrt{500}\mathrm{SD}({}_0L).$$

So, $500(7995.9459 - 2.6935939P) + 0.322036(10000 + 13.095P) = 100000$, and this gives $P = 2905.744$.

(c) The net premium is $10000.P_{60:\overline{30}|} = 10000 \left(\frac{1}{\ddot{a}_{60:\overline{30}|}} - d\right) = 3011.8744$.

Starting with ${}_0V = 0$, Fackler's accumulation formula gives

$${}_1V = \frac{3011.8744 \times 1.06 - 138}{1 - 0.0138} = 3097.32997$$

$${}_2V = \frac{(3011.8744 + 30977.32997) \times 1.06 - 150}{1 - 0.015} = 6422.08791$$

Of course ${}_3V = 10000$.

(d) Let the annual premium be G. A simple calculation gives

k	1.2^{-k}	${}_kp_{60}$	$1.2^{-(k+1)}{}_kp_{60}$	$1.2^{-k}{}_kp_{60}$
0	1	1	0.833333	1
1	0.83333333	0.9862	0.684861	0.821833
2	0.69444444	0.971407	0.562157	0.674588

The APV of the premium, discounted using 20% interest, is

$$(1 + 0.821833 + 0.674588)G = 2.496421G.$$

Then we compute the profit vector:

$$\begin{aligned}
\mathrm{Pr}_0 &= -30 \\
\mathrm{Pr}_1 &= 1.08G - 0.0138 \times 10100 - 0.9862 \times 3097.32997 = 1.08G - 3193.9668 \\
\mathrm{Pr}_2 &= 1.08(3097.32997 + 0.97G) - 0.015 \times 10100 - 0.985 \times 6422.08791 \\
&= 1.0476G - 3132.14022 \\
\mathrm{Pr}_3 &= 1.08(6422.08791 + 0.97G) - 10000 = 1.0476G - 3064.14506
\end{aligned}$$

The NPV of the project is

$$-30 + 0.833333\mathrm{Pr}_1 + 0.684861\mathrm{Pr}_2 + 0.562157\mathrm{Pr}_3 = 2.206376G - 6559.2501.$$

The profit margin is $\dfrac{2.206376G - 6559.2501}{2.496421G} = 0.15$. On solving, we get $G = 3580.55$.

16. We start with Pr_5. The death rate is $q_{x+4} = 0.017$.

$$\mathrm{Pr}_5 = -217 + 1.05\ {}_4V^Z = 0,$$

giving ${}_4V^Z = 206.66667$. Then by $q_{x+3} = 0.016$,

$$\mathrm{Pr}_4 = -55 + 1.05\ {}_3V^Z - 0.984\ {}_4V^Z \geq 0$$
$${}_3V^Z \geq (0.984 \times 206.66667 + 55)/1.05 = 246.0571429$$

So we get ${}_3V^Z = 246.0571429$ which makes $\mathrm{Pr}_3 = 0$. Then by $q_{x+2} = 0.015$,

$$\mathrm{Pr}_3 = 94 + 1.05\ {}_2V^Z - 0.985\ {}_3V^Z \geq 0$$
$${}_2V^Z \geq (0.985 \times 246.0571429 - 94)/1.05 = 141.3012245$$

So we get ${}_2V^Z = 141.3012245$ which makes $\mathrm{Pr}_3 = 0$. Then by $q_{x+1} = 0.014$,

$$\mathrm{Pr}_2 = 229 + 1.05\ {}_1V^Z - 0.986\ {}_2V^Z \geq 0$$
$${}_1V^Z \geq (0.986 \times 141.3012245 - 229)/1.05 = -85.4066$$

So we get ${}_1V^Z = 0$. This ${}_1V^Z$ makes $\mathrm{Pr}_2 = 229 - 0.986 \times 141.3012245 = 89.677$.

Finally we go to ${}_0V^Z$. Since $q_x = 0.013$,

$$\mathrm{Pr}_1 = 450 + 1.05{}_0V^Z - 0.987{}_1V^Z \geq 0$$
$${}_0V^Z \geq -450/1.05$$

So we get ${}_0V^Z = 0$. This ${}_0V^Z$ makes $\mathrm{Pr}_1 = 450$. The value of Pr_0 is unchanged as ${}_0V^Z = 0$.

The revised profit vector is $(-329,\ 450,\ 89.677,\ 0,\ 0,\ 0)'$.

17. (a) The expected profits are

$\mathrm{Pr}_0 = -0.3 \times 64 = -19.2$
$\mathrm{Pr}_1 = 64 \times 1.04 - (10000 + 10) \times 0.005 = 16.51$
$\mathrm{Pr}_2 = 0.95 \times 64 \times 1.04 - (10000 + 10) \times 0.007 = -6.838$
$\mathrm{Pr}_3 = 0.95 \times 64 \times 1.05 - (10000 + 10) \times 0.008 = -16.24$
$\mathrm{Pr}_4 = 0.95 \times 128 \times 1.05 - (10000 + 10) \times 0.009 = 37.59$
$\mathrm{Pr}_5 = 0.95 \times 128 \times 1.05 - (10000 + 10) \times 0.01 = 27.58$

The survival probabilities are found to be

k	${}_kp_{60}$
1	0.995
2	0.988035
3	0.98013072
4	0.971309544
5	0.961596448

The profit signature is

$$\mathbf{\Pi} = (-19.2,\ 16.51, -6.80381, -16.0457,\ 36.84311,\ 26.78872)'.$$

(b) The NPV is

$$-19.2+\frac{16.51}{1.1}-\frac{6.80381}{1.1^2}-\frac{16.0457}{1.1^3}+\frac{36.84311}{1.1^4}+\frac{26.78872}{1.1^5}=19.929.$$

(c) We set ${}_5V^Z=0$.

$\mathrm{Pr}_5 = 0.95\times 128\times 1.05-(10000+10)\times 0.01=27.58$

Since Pr_4 and Pr_5 are greater than 0, ${}_5V^Z={}_4V^Z={}_3V^Z=0$.

We look at $\mathrm{Pr}_3: -16.24+1.05{}_2V^Z=0\Rightarrow {}_2V^Z=15.466667$.

Then ${}_1V^Z=\max\left(\dfrac{15.4666667\times 0.993+6.838}{1.04},0\right)=\max(21.34269,\, 0)=21.34269.$

Then ${}_0V^Z=\max\left(\dfrac{21.34269\times 0.995-16.51}{1.04},0\right)=\max(4.54421,\, 0)=4.54421.$

The revised profit vector is given by

$\mathrm{Pr}_0=-19.2-4.54421=-23.7442$

$\mathrm{Pr}_1=\mathrm{Pr}_2=\mathrm{Pr}_3=0, \mathrm{Pr}_4=37.59, \mathrm{Pr}_5=27.58$

The revised profit signature is

$$\Pi=(-23.7442,\, 0,\, 0,\, 0,\, 36.84311,\, 26.78872)'.$$

(d) The NPV is now

$$-23.7442+\frac{36.84311}{1.1^4}+\frac{26.78872}{1.1^5}=18.054.$$

Using a financial calculator, the IRR for the policy is found to be 25.15%.

18. (a) (i) From the Illustrative Life Table, we can construct the following table:

x	p_x	q_x	k	${}_kp_{50}$
50	0.99408	0.00592	1	0.99408
51	0.99358	0.00642	2	0.987698
52	0.99303	0.00697	3	0.98081375
53	0.99242	0.00758	4	0.97337918

$\mathrm{Pr}_0=-80-50=-130$
$\mathrm{Pr}_1=(50+660-14)\times 1.07-100000q_{50}-50p_{50}=103.016$
$\mathrm{Pr}_2=(50+660-14)\times 1.07-100000q_{51}-50p_{51}=53.041$
$\mathrm{Pr}_3=(50+660-14)\times 1.07-100000q_{52}-50p_{52}=-1.9315$
$\mathrm{Pr}_4=(50+660-14)\times 1.07-100000q_{53}=-13.28$

(ii) The profit signature is

$$(-130,\ 103.016,\ 53.041 \times 0.99408, -1.9315 \times 0.987698, -13.28 \times 0.98081375)'$$
$$= (-130,\ 103.016,\ 52.726997, -1.907739, -13.025207)'$$

(iii) $\text{NPV} = -130 + \dfrac{103.016}{1.1} + \dfrac{52.726997}{1.1^2} - \dfrac{1.907739}{1.1^3} - \dfrac{13.025207}{1.1^4} = -3.1028$

(b) $\text{Pr}_4 = ({}_3V + 660 - 14) \times 1.07 - 100000q_{53} = 0 \Rightarrow {}_3V = 62.4112$

$\text{Pr}_3 = ({}_2V + 660 - 14) \times 1.07 - 100000q_{52} - 62.4112p_{52} = 0 \Rightarrow {}_2V = 63.323546$

$\text{Pr}_2 = ({}_1V + 660 - 14) \times 1.07 - 100000q_{51} - 63.323546p_{51} = 0 \Rightarrow {}_1V = 12.800942$

$\text{Pr}_1 = ({}_0V + 660 - 14) \times 1.07 - 100000q_{50} - 12.800942p_{50} = 0 \Rightarrow {}_0V = -80.836$

and hence we set ${}_0V = 0$.

(c) Using the zeroized reserves in (b), the profits at time 4, 3 and 2 would be zero, and profit at time 1 would be positive:

$\text{Pr}_1 = (660 - 14) \times 1.07 - 100000q_{50} - 12.800942p_{50} = 86.49484$

$\text{Pr}_0 = -80$

The NPV would be $-80 + 86.49487/1.1 = -1.3683$.

Exam ALTAM: General Information

Before you proceed to the mock tests, we would like to remind you a few things:

First, according to the SoA:

> *"The ALTAM Exam is a three-hour exam consisting of six questions, worth a total of 60 points.*
>
> *The exam will be taken at Prometric testing centers. The questions will be displayed on the computer. Candidates will also be provided with an Excel Workbook. The Excel Workbook will contain one question. It will also contain tables and formulas that may be used for any of the six questions.*
>
> *Five of the questions are to be answered in pen in exam answer booklets provided by Prometric. For these questions candidates may use the Excel Tables, and may use Excel for calculations, but only the answers provided in the exam answer booklet will be graded.*
>
> *One of the questions will be answered in the Excel Workbook. For this question, only the information provided in the Excel Workbook will be graded. Candidates may use the scratch paper booklet for rough notes, but nothing written on the scratch paper will be graded.*
>
> *At the end of the exam candidates will upload their Excel Workbook for grading the Excel question and will submit their exam answer booklets to Prometric to forward to the SOA for grading the questions answered in the answer booklet.*
>
> *Knowledge of the FAM Exam material is assumed for the ALTAM Exam."*

When you work on the mock exams, you may also use the Excel workbook that contain tables and formulas. You can download the workbook from the hyperlink in the Exam ALTAM Exam syllabus. Some functions in Excel such as normsdist and sumproduct can be quite useful when you work on questions that involve the Black-Scholes formula or the standard and ultimate life table. You can also implement Euler's method and do profit and NPV calculations quite easily by setting up tables. However, you need to make sures you show formulas in your working and copy the tables into your answer booklet if such questions are to be answered in pen but not in the Excel Workbook that would be uploaded.

Second, according to the SoA, there is an approximate distribution of topics:

Topic	Our manual	Percentage in Exam	Points (out of 60)
1. Survival Models for Multiple State Contingent Cashflows	Chapters 2, 5, 6	10 – 20%	6 to 12
2. Premium and Policy Valuation for Long-term State-dependent Coverages	Chapters 0, 3, 4, 5, 6	12 – 20%	6 to 12
3. Joint Life Insurance and Annuities	Chapters 2, 5	8 – 16%	4.8 to 9.6
4. Profit Analysis	Chapter 1	10 – 20%	6 to 12
5. Pension Plans and Retirement Benefits	Chapters 7, 8	10 – 18%	6 to 10.8
6. Universal Life Insurance	Chapter 9	10 – 18%	6 to 10.8
7. Embedded Options in Life Insurance and Annuity Products	Chapter 10	10 – 18%	6 to 10.8

Don't expect that there would be no question coming from the topic of (say) pension. If you miss one single topic, you can lose quite a lot of points because you will certainly have to skip a question.

Finally, the SoA says that:

> *Unless specified otherwise within the examination question, the following assumptions should be made:*
>
> (i) *The force of interest is constant and is greater than 0.*
>
> (ii) *Future lifetimes are independent.*
>
> (iii) *All lives in a question follow the same mortality table.*
>
> (iv) *Expenses are payable at the start of each period.*
>
> (v) *Premiums are payable throughout the term of the contract.*
>
> (vi) *Expenses (including commissions) that are expressed as a percent of premium are payable when the corresponding premium is payable, and end when the premiums are no longer payable.*
>
> (vii) *Confidence intervals are 2-sided, and are based on the normal approximation.*

Please follow the conventions mentioned above when you work on the mock tests.

Good luck!

Mock Exam 1

****BEGINNING OF EXAMINATION****

1. (*18 points*) This is an Excel worksheet question. Please refer to the **Excel workbook for mock exams**.

2. (*10 points*)

   (a) (*2 points*) Define the meaning of $1000\bar{A}^2_{xy}$ in words.

   For two independent lives (30) and (40), you are given that for (30), $\mu_{30+t} = 0.02$, and that for (40), $\mu_{40+t} = 0.03$. The constant force of interest is 4%.

   (b) (*2 points*) Calculate $1000\bar{A}^2_{30:40}$.

   (c) (*2 points*) For the policy in (b), annual premium is payable continuously until the second death. Calculate the annual premium rate and outline the main deficiency of such premium paying scheme.

   (d) (*4 points*) If the two lives are dependent, and the true underlying model is an exponential common shock model with the common shock component following an exponential distribution with rate 0.01, recalculate $1000\bar{A}^2_{30:40}$. Assuming that there is no payment if both lives die at the same time, and that the marginal forces of mortality are the same.

3. (*11 points*) A life insurance company issues a three-year term insurance with a death benefit of 10,000 to a male aged 65 exact. The single premium payable at outset on the policy is 250. The death benefit is paid at the end of the year of death.

The company uses the following basis to profit test the policy:

Mortality:	$q_{65} = 0.0060$, $q_{66} = 0.0066$, $q_{67} = 0.0075$
Interest earned on cash flow and reserves:	5%
Pre contractual expense:	5% of the single premium for commission plus 40
Renewal expenses:	10 and 15, payable at the beginning of the second and the third policy year if the policyholder survives
Risk discount rate:	6.5%

In addition, the company establishes reserves on the policy, where

$$_tV = 50(3-t), \text{ for } t = 1, 2, 3$$

and $_0V = 0$, calculated from a conservative mortality and interest assumption.

(a) (*8 points*) Calculate the net present value of the expected profits on the policy:

(i) allowing for reserves;

(ii) ignoring reserves.

(b) (*2 points*) Comment on the reason for the difference in the two values calculated in (a).

(c) (*1 point*) Calculate the profit margin for the policy allowing for reserves.

4. (*7 points*) An employer offers retiree health benefits to his employees. You are given:

- $B(y,t)$ is the annual health premium payable at time t, for a life aged y at time t.
- The valuation interest rate is $i = 0.05$.
- The inflation rate for the health care costs is $B(y,t+1)/B(y,t) - 1 = 0.03$ for all y and t.
- Health care costs increases with age according to $B(y+1,t)/B(y,t) - 1 = 0.015$ for all y and t.
- $B(62,\ 0) = 4500$
- Pre-retirement decrements follow the Standard Service Table.
- Retirements occur at midyear for age range (62, 63), (63, 64) and (64, 65).
- The following annuity values under an interest rate of 0.43522% for a retiree.

x	62	62.5	63	63.5	64	64.5	65
$\ddot{a}_x$	24.3514	23.9817	23.5830	23.1855	22.7891	22.3940	22.0003

(a) (*3 points*) Calculate $AVTHB_{62}$, the actuarial value of the total health benefit for an active employee at age 62 at time 0.

(b) (*4 points*) Calculate the accrued liability and normal contribution for an employee aged 62, with 17 years of service, by assuming linear accrual to each retirement age.

5. (*7 points*)

(a) (*2 points*) Explain in words the difference between Type A and Type B universal life insurance policies.

(b) (*3 points*) For a Type A universal life insurance policy, you are given:

(i) The face amount of the policy is \$60,000.

(ii) The account value on December 31, 2013 was \$30,000.

(iii) On January 1, 2014, a premium of \$2,500 was made. No other premiums were made in 2014.

(iv) The total expense charge and the cost of insurance deducted on January 1, 2014 were \$170 and \$250, respectively.

(v) The credited interest rate in 2014 was 8% per annum effective.

(vi) The corridor factor requirement is 2.5.

(vii) Death occurred in 2014 and a death benefit was payable at the end of 2014.

Compute the amount of the death benefit.

(c) (*2 points*) Suppose that the universal life insurance policy in (b) is Type B instead. Recalculate the amount of the death benefit.

6. (*7 points*) The risk of embedded put options in equity-linked insurance can be mitigated by internally hedging the risk of the sold options using delta-hedging, or by purchasing a portfolio of put options from the options market that replicate the put options sold to policyholders.

   (a) (*2 points*) Explain why it is not enough for the insurer to simply hold a reserve with value that is equal to the cost of the put options.

   (b) (*5 points*) Discuss some of the cons of delta-hedging the embedded put options and purchasing a portfolio of put options from the options market.

**** END OF EXAMINATION ****

Solutions to Mock Test 1

1. [Chapter 2 + 3] This is an Excel worksheet question. Please refer to the **Excel workbook for mock exams**.

2. [Chapter 4 + 5]

(a) It is the expected present value of a fully continuous insurance for a benefit of 1000, written on 2 lives aged x and y, where the sum is paid on the death of x only if life aged x dies after life aged y.

(b) $1000\int_0^\infty e^{-0.04t}{}_tq_{40\,t}p_{30}\mu_{30+t}\mathrm{d}t$

$= 1000\int_0^\infty e^{-0.04t}(1-e^{-0.03t})e^{-0.02t}0.02\mathrm{d}t$

$= 20\int_0^\infty (e^{-0.06t}-e^{-0.09t})\mathrm{d}t = 20\left(\frac{1}{0.06}-\frac{1}{0.09}\right)$

$= \frac{1000}{9}$

(c) $\bar{a}_{30:40} = \int_0^\infty e^{-0.04t}{}_tp_{30\,t}p_{40}\mathrm{d}t = \int_0^\infty e^{-0.04t}e^{-0.02t}e^{-0.03t}\mathrm{d}t = \frac{1}{0.09}$

$\bar{a}_{30} = \frac{1}{0.04+0.02} = \frac{1}{0.06}$, $\bar{a}_{40} = \frac{1}{0.04+0.03} = \frac{1}{0.07}$

$\bar{a}_{\overline{30:40}} = \frac{1}{0.06}+\frac{1}{0.07}-\frac{1}{0.09} = \frac{1250}{63}$

The premium required is $(1000/9)/(1250/63) = 5.6$.

If the life aged 30 dies first, then the policy would cease without any future benefit, but yet the life aged 40 is still expected to pay premiums so long as he or she survives. The lapse rate would be very high.

(d) $T_{30}* \sim \exp(0.01)$, and $T_{40}* \sim \exp(0.02)$

Let State 0 be the state when both are alive, State 1 be the state when (40) is dead but (30) is alive, State 2 be the state when (30) is dead but (40) is alive, and State 3 be the state when both are dead.

$$ {}_tp^{01}_{30:40} = \int_0^t {}_sp^{00}_{30:40}\mu^{01}_{30+s:40+s}{}_{t-s}p^{11}_{30+s}\mathrm{d}s = \int_0^t e^{-0.04s}0.02e^{-0.02(t-s)}\mathrm{d}s $$

$$ = 0.02e^{-0.02t}\int_0^t e^{-0.02s}\mathrm{d}s = e^{-0.02t}(1-e^{-0.02t}) $$

The density of the time to enter 3 through the route "0 to 1 to 3" is

$$ {}_tp^{01}_{30:40}(\mu_{40+t}+\lambda) = 0.02(e^{-0.02t}-e^{-0.04t}). $$

The EPV is

$$ 1000\int_0^\infty e^{-0.04t}0.02(e^{-0.02t}-e^{-0.04t})\mathrm{d}t = 20\left(\frac{1}{0.06}-\frac{1}{0.08}\right) = 83.33. $$