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Study Manual for Exam FAM-S

2nd Edition

Sam A. Broverman, PhD, ASA

An SOA Exam

ACTEX Learning

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2nd Edition

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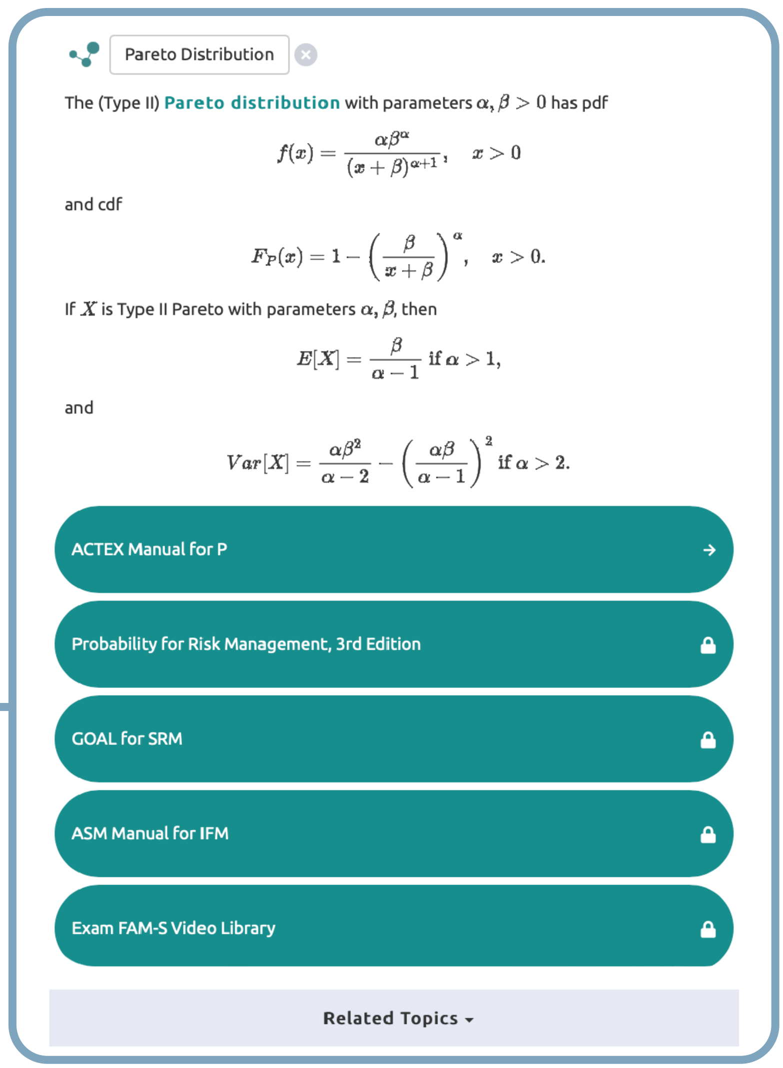

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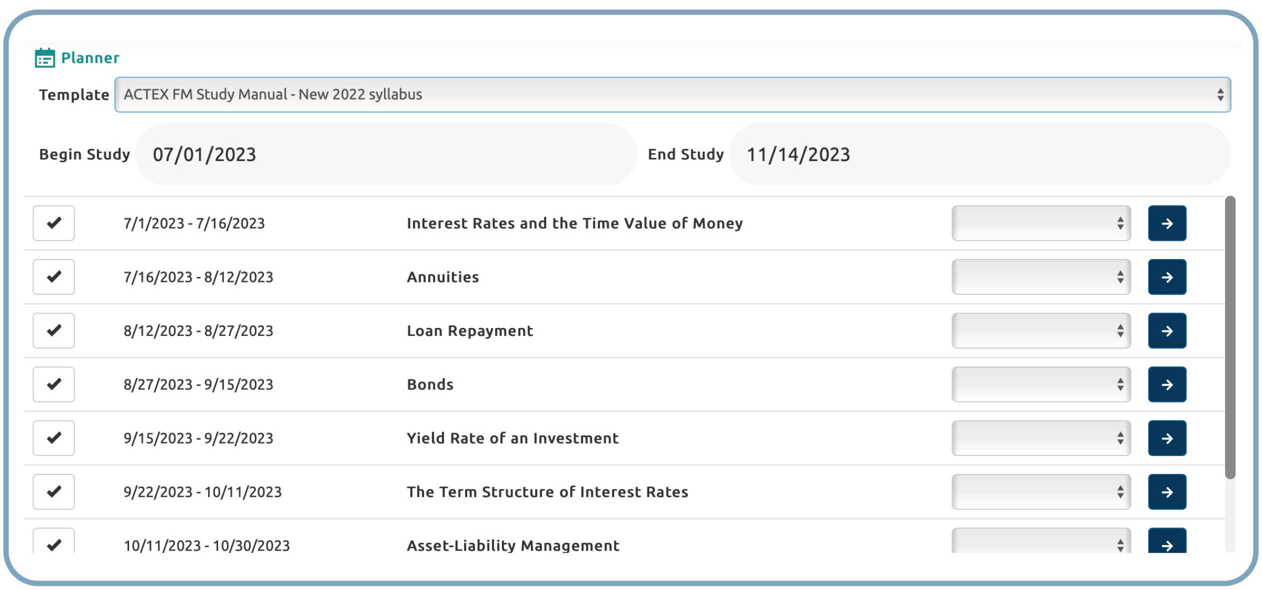

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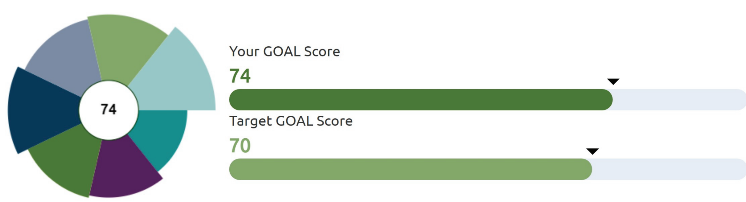

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87 G. Pension Plans and Retirement Benefits Weighting: 15%		

See key areas where you can improve.

A. Multiple State Models

A1. Multiple State Probabilities

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Core	38	22 / 48
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Created	Last Accessed	Completed	Mode	Categories	Questions	Status	Action
03/24/2023	03/24/2023	03/24/2023	Quiz	G. Pension Plans and Retirement Benefits	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	C1. Joint Life/Last Survivor Probabilities, C2. Contingent Probabilities, C3. Expectation and Variance, C6. ML Models with Transition Intensities, C4. ML Models: Premiums and Reserves (Discrete), C5. ML Models: Premiums and Reserves (Continuous)	15	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	F. Embedded Options and Equity-Linked Insurance	20	Complete	Review
03/24/2023	03/24/2023	03/24/2023	Quiz	D. Profit Analysis	10	Complete	Review
09/19/2022	09/19/2022	09/19/2022	Quiz	G4. Pension Plans and Retirement Benefits (Written Answers)	2	Complete	Review

Quickly return to previous sessions.

Contents

INTRODUCTORY COMMENTS

The FAM exam is divided almost equally into FAM-S and FAM-L topics. This study manual is designed to help in the preparation for the FAM-S part of the Society of Actuaries FAM Exam.

The first part of this manual consists of a summary of notes, illustrative examples and problem sets with detailed solutions. The second part consists of 5 practice exams. The SOA exam syllabus for the FAM exam indicates that the exam is 3.5 hours in length with 34 multiple choice questions. The practice exams in this manual each have 17 questions, reflecting the fact that FAM-S is 50% of the full FAM exam. The appropriate time for the 17 question FAM-S practice exams in this manual is one hour and forty-five minutes.

The level of difficulty of the practice exam questions has been designed to be similar to those on past exams covering the same topics. The practice exam questions are not from old SOA exams.

I have attempted to be thorough in the coverage of the topics upon which the exam is based, and consistent with the notation and content of the official references. I have been, perhaps, more thorough than necessary on reviewing maximum likelihood estimation.

Because of the time constraint on the exam, a crucial aspect of exam taking is the ability to work quickly. I believe that working through many problems and examples is a good way to build up the speed at which you work. It can also be worthwhile to work through problems that you have been done before, as this helps to reinforce familiarity, understanding and confidence. Working many problems will also help in being able to more quickly identify topic and question types. I have attempted, wherever possible, to emphasize shortcuts and efficient and systematic ways of setting up solutions. There are also occasional comments on interpretation of the language used in some exam questions. While the focus of the study manual is on exam preparation, from time to time there will be comments on underlying theory in places that I feel those comments may provide useful insight into a topic.

The notes and examples are divided into 31 sections of varying lengths, with some suggested time frames for covering the material. There are almost 180 examples in the notes and over 440 exercises in the problem sets, all with detailed solutions. The 5 practice exams have 17 questions each, also with detailed solutions. Some of the examples and exercises are taken from previous SOA exams. Some of the problem set exercises are more in depth than actual exam questions, but the practice exam questions have been created in an attempt to replicate the level of depth and difficulty of actual exam questions. In total there are almost 700 examples/problems/sample exam questions with detailed solutions. ACTEX gratefully acknowledges the SOA for allowing the use of their exam problems in this study manual.

I suggest that you work through the study manual by studying a section of notes and then attempting the exercises in the problem set that follows that section. The order of the sections of notes is the order that I recommend in covering the material, although the material on short-term insurance pricing and reserving in Sections 27 to 30 and option pricing in Section 31 is independent of the other material on the exam. The order of topics in this manual is not the same as the order presented on the exam syllabus.

It has been my intention to make this study manual self-contained and comprehensive, however it is important to be familiar with original reference material on all topics.

While the ability to derive formulas used on the exam is usually not the focus of an exam question, it is useful in enhancing the understanding of the material and may be helpful in memorizing formulas. There may be an occasional reference in the review notes to a derivation, but you are encouraged to review the official reference material for more detail on formula derivations.

In order for the review notes in this study manual to be most effective, you should have some background at the junior or senior college level in probability and statistics. It will be assumed that you are reasonably familiar with differential and integral calculus. The prerequisite concepts to modeling and model estimation are reviewed in this study manual. The study manual begins with a detailed review of probability distribution concepts such as distribution function, hazard rate, expectation and variance. Of the various calculators that are allowed for use on the exam, I am most familiar with the BA II PLUS. It has several easily accessible memories. The TI-30X IIS has the advantage of a multi-line display. Both have the functionality needed for the exam.

There is a set of tables that has been provided with the exam in past sittings. These tables consist of some detailed description of a number of probability distributions along with tables for the standard normal and chi-squared distributions. The tables can be downloaded from the SOA website www.soa.org.

If you have any questions, comments, criticisms or compliments regarding this study manual, please contact the publisher ACTEX, or you may contact me directly at the address below. I apologize in advance for any errors, typographical or otherwise, that you might find, and it would be greatly appreciated if you would bring them to my attention. ACTEX will be maintaining a website for errata that can be accessed from https://actexlearning.com/errata. It is my sincere hope that you find this study manual helpful and useful in your preparation for the exam. I wish you the best of luck on the exam.

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Section 7

Mixture Of Two Distributions

The material in this section relates to Section 4.2.3 of “Loss Models”. The suggested time frame for this section is 2 hours. The topic of distribution mixtures is not mentioned in the learning objectives for the FAM exam but all of Chapter 4 of “Loss Models” is listed in the reference reading.

7.1 Mixture of Two Distributions

We begin with a formal algebraic definition of how a mixture of two distributions is constructed, and later we will look at how a mixture distribution is described by general reasoning.

Given random variables X_1 and X_2, with pdf's or pf's $f_{X_1}(x)$ and $f_{X_2}(x)$, and given $0 < a < 1$, we construct the random variable Y with pdf

$$\boldsymbol{f_Y(y) = a \times f_{X_1}(y) + (1-a) \times f_{X_2}(y)} \tag{7.1}$$

Y is called a **mixture distribution** or a two-point mixture of the distributions of X_1 and X_2.

The two-point mixture random variable Y can also be defined in terms of the cdf,

$$\boldsymbol{F_Y(y) = a \times F_{X_1}(y) + (1-a) \times F_{X_2}(y)} \tag{7.2}$$

X_1 and X_2 are the **component distributions** of the mixture, and the factors a and $1-a$ are referred to as **mixing weights**. It is important to understand that we are **not adding** aX_1 and $(1-a)X_2$, Y **is not** a sum of random variables. Y is defined in terms of a pdf (or cdf) that is a weighted average of the pdf's (or cdf's) of X_1 and X_2. We are adding af_{X_1} and $(1-a)f_{X_2}$ to get f_Y.

Example 7.1. As a simple illustration of a mixture distribution, consider two bowls. Bowl A has 5 balls with the number 1 on them and 5 balls with the number 2 on them, and bowl B has 3 balls with the number 1 and 7 balls with the number 2. Let X_1 denote the number on a ball randomly chosen from bowl A, and let X_2 denote the number on a ball randomly chosen from bowl B. The probability functions of X_1 and X_2 are
$f_{X_1}(1) = f_{X_1}(2) = 0.5$ and $f_{X_2}(1) = 0.3$, $f_{X_2}(2) = 0.7$.

Suppose we create the mixture distribution with mixing weights $a = 0.5$ and $1-a = 0.5$. The mixture distribution Y has probability function $f_Y(1) = 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4$, $f_Y(2) = 0.5 \times 0.5 + 0.5 \times 0.7 = 0.6$.

Note that the outcomes (ball numbers) of the mixture distribution Y come from the possible outcomes of the component distributions X_1 and X_2. □

An alternative way of looking at this mixture distribution is by means of conditioning on a "parameter". This will be important when we look at continuous mixing. The parameter approach to describe the mixture distribution in Example 7.1 is as follows.

Suppose that a fair coin is tossed. If the toss is a head, a ball is chosen at random from bowl A and if the toss is a tail, a ball is chosen at random from bowl B. We define the random variable Z to be the number on the ball. We will see that Z has the same distribution as the mixture distribution labeled Y above.

The random variable Z can be interpreted as follows. Consider the 2-point random variable Θ, for which Θ =Bowl A if the coin toss is a head, and Θ =Bowl B if the toss is a tail. Then $P(\Theta = A) = P(\Theta = B) = .5$ (since the coin is fair). Θ is used to indicate which bowl the ball will be chosen from depending on the outcome of the coin toss.

If the toss is a head, the bowl is A, and then Z has the X_1 distribution for the number on the ball, so $f_{X_1}(1) = P(Z = 1|\Theta = A) = 0.5$ and $f_{X_1}(2) = P(Z = 2|\Theta = A) = 0.5$. In a similar way, if the toss is a tail, the bowl is B, and then Z has the X_2 distribution for the number on the ball, so $f_{X_2}(1) = P(Z = 1|\Theta = B) = 0.3$ and $f_{X_2}(2) = P(Z = 2|\Theta = B) = 0.7$.
Z is described as a combination of two conditional distributions based on the parameter Θ.

To find the overall, or unconditional distribution of Z, we use some basic rules of probability. Since Θ must be A or B, we can think of bowl B as the "complement" of bowl A, and then

$$\begin{aligned} P(Z = 1) &= P[(Z = 1) \cap (\Theta = A)] + P[(Z = 1) \cap (\Theta = B)] \\ &= P(Z = 1|\Theta = A) \times P(\Theta = A) + P(Z = 1|\Theta = B) \times P(\Theta = B) \\ &= 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4 \end{aligned}$$

We have used the rule $P(C) = P[C \cap D] + P[C \cap D'] = P(C|D) \times P(D) + P(C|D') \times P(D')$.

This shows that the distribution of Z is the same as the mixture distribution Y in Example 7.1. The mixing weights for the two bowls are the probabilities of the coin indicating bowl A or bowl B.

Language used on exam questions that identifies a mixture distribution
There is some typical language that is used in exam questions that indicates that a mixture of distributions is being considered. This language will be illustrated using the distributions involved in the bowl example.

Suppose that we are told that there are two types of individuals. Type A individuals have a mortality probability of .5 (and survival probability of .5) in the coming year, and Type B individuals have a mortality probability of .3 in the coming year. In a large group of these individuals, 50% are Type A and 50% are Type B. An individual is chosen at random from the group. We want to find this randomly chosen individual's mortality probability.

We can use the usual rules of conditional probability to formulate this probability, just as above:

$$\begin{aligned} P(\text{dying this year}) &= P(\text{dying} \cap \text{Type A}) + P(\text{dying} \cap \text{Type B}) \\ &= P(\text{dying}|\text{Type A}) \times P(\text{Type A}) + P(\text{dying} \cap \text{Type B}) \times P(\text{Type B}) \\ &= 0.5 \times 0.5 + 0.5 \times 0.3 = 0.4. \end{aligned}$$

This is exactly $f_Y(1) = f_{X_1}(1) \times a + f_{X_2}(1) \times (1-a)$ where "1" corresponds to the event of dying within the year. Note that the phrase "50% are Type A" is interpreted as meaning that if an individual is chosen from the large group, the probability of being Type A is 0.5. This is language that is often used in exam questions to indicate a mixture.

Mixture of a discrete and a continuous distribution
In Section 2 of this study guide we looked at "mixed distributions". In that section, a mixed distribution referred to a random variable that was continuous on part of its probability space and also had one or more discrete points. The concept of mixed distribution just introduced in this section can be used to describe the Section 2 type of mixed distribution. The following example uses the example from Section 2 to illustrate this.

Example 7.2. Suppose that X is defined in the following piecewise way: X has a discrete point of probability of .5 at $X = 0$, and on the interval (0,1) X is a continuous random variable with density function $f(x) = x$ for $0 < x < 1$, and X has no density or probability elsewhere. Show that this random variable can be described as a mixture of two distributions with appropriate definitions for component distributions X_1, X_2 and mixing weights a and $1-a$.

Solution.

Let $X_1 = 0$ be a constant (not actually a random variable, but is sometimes called a degenerate random variable), so that $f_{X_1}(0) = P(X_1 = 0) = 1$ and $f_{X_1}(x) = 0$ for $x \neq 0$.

Let X_2 be continuous on (0,1) with pdf $f_{X_2}(x) = 2x$.

With mixing weights of $a = 0.5$ and $1-a = 0.5$, using the definition of the mixture of two distributions, we have the mixed random variable Y satisfying

$f_Y(0) = a \times f_{X_1}(0) + (1-a) \times f_{X_2}(0) = (0.5 \times 1) + 0 = 0.5$, and
$f_Y(x) = a \times f_{X_1}(x) + (1-a) \times f_{X_2}(x) = 0.5 \times 0 + 0.5 \times 2x = x$ for $0 < x < 1$.
Y has the same distribution as the original X. □

We should be a little careful about the situation in Example 7.2. When we are mixing discrete probability points with a continuous density, for any particular discrete point we always assign a probability of 0 at that point for any continuous component distribution. For instance, in Example 7.2, if X_1 was at the single point 0.4, then $f_{X_1}(0.4) = 1$ and $f_{X_1}(x) = 0$ elsewhere, and for the mixture distribution, then $f_Y(0.4) = 0.5 \times 1 = 0.5$ (we do not add $0.5 \times f_{X_2}(0.4)$).

Some important relationships for mixture distributions
We have already seen the defining relationships

$$\boldsymbol{f_Y(y) = af_{X_1}(y) + (1-a)f_{X_2}(y)}$$

and

$$\boldsymbol{F_Y(y) = aF_{X_1}(y) + (1-a)F_{X_2}(y)}.$$

We can interpret these relationships by saying that the pdf and cdf of Y are weighted averages of the component pdf's and cdf's. This weighted average interpretation can be applied to a number of other distribution related quantities.

- if C is any event related to Y, then $P(C) = a \times P_{X_1}(C) + (1-a) \times P_{X_2}(C)$; (7.3)
  $P_{X_1}(C)$ is the event probability based on random variable X_1, and the same for X_2
- If g is any function (that doesn't involve the parameters of Y), then (7.4)
  $E[g(Y)] = a \times E[g(X_1)] + (1-a) \times E[g(X_2)]$

The justification for this relationship follows from the form of the mixed density;

$$\begin{aligned} E[g(Y)] &= \int g(y)\, f_Y(y)\, dy = \int g(y) \times [a \times f_{X_1}(y) + (1-a) \times f_{X_2}(y)]\, dy \\ &= a \times \int g(y) \times f_{X_1}(y)\, dy + (1-a) \times \int g(y) \times f_{X_2}(y)\, dy \\ &= a \times E[g(X_1)] + (1-a) \times E[g(X_2)] \end{aligned}$$

Some of the important examples of these weighted average relationships are:

Interval Probability: the event C is $c < Y \le d$

$$\boldsymbol{P(c < Y \le d) = a \times P(c < X_1 \le d) + (1-a) \times P(c < X_2 \le d)} \quad (7.5)$$

kth moment of Y: the function g is $g(Y) = Y^k$

$$\boldsymbol{E[Y^k] = a \times E[X_1^k] + (1-a) \times E[X_2^k]} \quad (7.6)$$

7.2 Formulating a Mixture Distribution as a Combination of Conditional Distributions

The relationships above can also be formulated as applications of probability rules involving conditional probability and conditional expectation.

(i) For any random variable Y and event D,

$$\boldsymbol{f_Y(y) = f(y|D) \times P(D) + f(y|D') \times P(D')} \quad (7.7)$$

(ii) For any events C and D,

$$\boldsymbol{P[C] = P[C \cap D] + P[C \cap D'] = P[C|D] \times P[D] + P[C|D'] \times P[D'].} \quad (7.8)$$

(iii) For any random variable Y and any event D,

$$\boldsymbol{E[Y] = E[Y|D] \times P[D] + E[Y|D'] \times P[D']} \quad (7.9)$$

We can define the random variable Θ to be 1 or 2, indicating whether or not event D has occurred, so $\Theta = 1 \equiv D$ has occurred, and $\Theta = 2 \equiv D'$ has not occurred.

We can think of X_1 as the conditional distribution of Y given $\Theta = 1$ (event D) and we can think of X_2 as the conditional distribution of Y given $\Theta = 2$ (event D'), so that

$$f(y|D) = f(y|\Theta = 1) = f_{X_1}(y) \text{ and } f(y|D') = f(y|\Theta = 2) = f_{X_2}(y).$$

The cdf, pdf and expectation of Y are adaptations of the conditional probability rules above. For instance, the mean of the mixed distribution is the "mixture of the means" of the component distributions.

$$E[Y] = E[Y|\Theta = 1] \times P[\Theta = 1] + E[Y|\Theta = 2] \times P[\Theta = 2] = E[X_1] \times a + E[X_2] \times (1-a).$$

Example 7.3. A collection of insurance policies consists of two types. 25% of policies are Type 1 and 75% of policies are Type 2. For a policy of Type 1, the loss amount per year follows an exponential distribution with mean 200, and for a policy of Type 2, the loss amount per year follows a Pareto distribution with parameters $\alpha = 3$ and $\theta = 200$. For a policy chosen at random from the entire collection of both types of policies, find the probability that the annual loss will be less than 100, and find the average loss.

Solution.

The two types of losses are the random variables X_1 and X_2.

X_1 has an exponential distribution with mean 200, so it has cdf $F_{X_1}(x) = 1 - e^{-x/200}$.

X_2 has cdf $F_{X_2}(x) = 1 - \left(\frac{\theta}{x+\theta}\right)^\alpha = 1 - \left(\frac{200}{x+200}\right)^3$.

These are the component distributions. The mixing weights are the proportions of policies of each type, so that $a = 0.25$ (the proportion of Type 1 policies) and $1 - a = 0.75$. The loss random variable Y of the randomly chosen policy has a distribution that is the mixture of X_1 and X_2. In this context, "randomly chosen" is taken to mean that the probabilities of choosing a Type 1 or Type 2 policy are the proportions of those policy types in the full collection of policies.

The cdf of Y is

$$\begin{aligned} F_Y(y) &= a \times F_{X_1}(y) + (1-a) \times F_{X_2}(y) \\ &= 0.25 \times (1 - e^{-y/200}) + 0.75 \times \left(1 - \left(\frac{200}{y+200}\right)^3\right). \end{aligned}$$

Then, $F_Y(100) = 0.626$.

The mean of Y is $0.25 \times E[X_1] + 0.75 \times E[X_2] = 0.25 \times 200 + 0.75 \times \frac{200}{3-1} = 125$. □

7.3 The Variance of a Mixed Distribution IS (usually) NOT the Weighted Average of the Variances

For a mixture of two random variables, the weighted average pattern applies to most quantities for the mixed distribution.

One important exception to this rule is the formulation of the **variance of the mixture**. If Y is any random variable, then we can formulate the variance of Y as $Var[Y] = E[Y^2] - (E[Y])^2$.

If Y is the mixture of X_1 and X_2 with mixing weights a and $1-a$, then we know that $E[Y] = a \times E[X_1] + (1-a) \times E[X_2]$ and $E[Y^2] = a \times E[X_1^2] + (1-a) \times E[X_2^2]$.

It is tempting to guess that $Var[Y]$ is the weighted average of $Var[X_1]$ and $Var[X_2]$.

This is **<u>not true</u>** in general (we will see a special case later for which it is true).

Example 7.4. Find the variance of the loss on the policy chosen at random in Example 7.3, and compare it to the weighted average of the variances of the two component loss distributions.

Solution.

The mean of Y is $0.25 \times E[X_1] + 0.75 \times E[X_2] = 0.25 \times 200 + 0.75 \times \frac{200}{3-1} = 125$,

and the second moment of Y is

$0.25 \times E[X_1^2] + 0.75 \times E[X_2^2] = 0.25 \times 2 \times 200^2 + 0.75 \times \frac{200^2 \times 2}{(3-1)(3-2)} = 50{,}000$, so

$Var[Y] = E[Y^2] - (E[Y])^2 = 34{,}375$.

$Var[X_1] = 40{,}000$ (variance of an exponential distribution is the square of the mean).

$Var[X_2] = \frac{200^2 \times 2}{(3-1)(3-2)} - 100^2 = 30{,}000$.

The weighted average of the two variances is $0.25 \times 40{,}000 + 0.75 \times 30{,}000 = 32{,}500$, which is not the variance of Y. □

Weighted average does not apply to percentiles in a mixture distribution
In Example 7.3, the 50th percentile of X_1 is m_1, the solution of the equation $F_{X_1}(m_1) = 1 - e^{-m_1/200} = 0.5$, so that $m_1 = 138.63$.

The 50th percentile of X_2 is m_2, the solution of $F_{X_2}(m_2) = 1 - \left(\frac{200}{m_2+200}\right)^3 = 0.5$, so that $m_2 = 51.98$.

The weighted average of the two 50th percentiles is $0.25 \times 138.63 + 0.75 \times 51.98 = 73.64$. The cdf of the mixed distribution Y is $F_Y(y) = 0.25 \times F_{X_1}(y) + 0.75 \times F_{X_2}(y)$, and $F_Y(73.64) = 0.53$, so 73.64 is not the 50th percentile of Y.

To find the 50th percentile, say m, of the mixed distribution in Example 7.3, we must solve the equation

$$\begin{aligned} .50 &= F_Y(m) = 0.25 \times F_{X_1}(m) + 0.75 \times F_{X_2}(m) \\ &= 0.25 \times (1 - e^{-m/200}) + 0.75 \times \left(1 - \left(\frac{200}{m+200}\right)^3\right). \end{aligned}$$

There is no algebraic solution, and m would have to be found by a numerical approximation method. The 50th percentile turns out to be about 65.7.

Section 7 Problem Set

Mixture Of Two Distributions

1. A portfolio of insurance policies is divided into low and high risk policies. 80% of the policies are low risk and the rest are high risk. The annual number of claims on a low risk policy has a Poisson distribution with a mean of .25 and the annual number of claims on a high risk policy has a Poisson distribution with a mean of 2. A policy is randomly chosen from the portfolio.

   (a) Find the mean and variance of the number of annual claims for the policy.

   (b) Find the probability that the policy has at most 1 claim in the coming year.

2. The distribution of a loss, X, is a two-point mixture:

   (i) With probability 0.8, X has a two-parameter Pareto distribution with $\alpha = 2$ and $\theta = 100$.

   (ii) With probability 0.2, X has a two-parameter Pareto distribution with $\alpha = 4$ and $\theta = 3000$.

   Calculate $Pr(X \leq 200)$.

   (A) 0.76 (B) 0.79 (C) 0.82 (D) 0.85 (E) 0.88

3. The random variable N has a mixed distribution:

   (i) With probability p, N has a binomial distribution with $q = 0.5$ and $m = 2$.

   (ii) With probability $1 - p$, N has a binomial distribution with $q = 0.5$ and $m = 4$.

   Which of the following is a correct expression for Prob$(N = 2)$?

   (A) $0.125p^2$
   (B) $0.375 + 0.125p$
   (C) $0.375 + 0.125p^2$
   (D) $0.375 - 0.125p^2$
   (E) $0.375 - 0.125p$

4. Y is a mixture of two exponential distributions, $f_Y(y) = \frac{1}{2}e^{-y} + \frac{1}{6}e^{-y/3}$.
   The random variable Z defined by the equation $Z = 2Y$.
   Z is a mixture of two exponentials. The means of those two exponential distributions are

   (A) 1 and 3
   (B) 1 and 6
   (C) 2 and 3
   (D) 2 and 6
   (E) 3 and 6

5. X_1 has a uniform distribution on the interval (0,1000) and X_2 has a uniform distribution on the interval (0,2000). Y is defined as a mixture of X_1 and X_2 with mixing weights of .5 for each mixture component. Find the pdf, cdf and median (50th percentile) of Y.

6. A population of people aged 50 consists of twice as many non-smokers as smokers. Non-smokers at age 50 have a mortality probability of .1 and smokers at age 50 have a mortality probability of .2. Two 50-year old individuals are chosen at random from the population.

   (a) Find the probability that at least one of them dies before age 51.

   (b) Suppose that the mortality probabilities for smokers and non-smokers remain the same at age 51. Find the mortality probability of a randomly chosen survivor at age 51 in this population.

7. Y is the mixture of an exponential random variable with mean 1 and mixing weight $\frac{2}{3}$, and an exponential distribution with mean 2 and mixing weight $\frac{1}{3}$.

   Find the pdf, cdf, mean, variance and 90th percentile of Y.

8. Which of the following statements are true?

   I. A mixture of two different exponentials with the mixture having a mean 2 has a heavier right tail than a single exponential distribution with mean 2.

   II. If $f_Y(y) = a_1 \times f_{X_1}(y) + a_2 \times f_{X_2}(y)$, where $0 < a_1 < 1$ and $0 < a_2 < 1$, then $e_Y(d) = a_1 \times e_{X_1}(d) + a_2 \times e_{X_2}(d)$.

9. You are given two independent estimates of an unknown quantity, μ:

   (i) Estimate A: $E(\mu_A)$= 1000 and $\sigma(\mu_A)$= 400

   (ii) Estimate B: $E(\mu_B)$= 1200 and $\sigma(\mu_B)$= 200

   Estimate C is a weighted average of the two estimates A and B, such that:

   $$\mu_C = w \times \mu_A + (1 - w) \times \mu_B$$

   Determine the value of w that minimizes $\sigma(\mu_C)$.

   (A) 0 (B) 1/5 (C) 1/4 (D) 1/3 (E) 1/2

10. You are given the claim count data for which the sample mean is roughly equal to the sample variance. Thus you would like to use a claim count model that has its mean equal to its variance. An obvious choice is the Poisson distribution. Determine which of the following models may also be appropriate.

    (A) A mixture of two binomial distributions with different means.

    (B) A mixture of two Poisson distributions with different means.

    (C) A mixture of two negative binomial distributions with different means.

    (D) None of A, B, or C

    (E) All of A, B, and C

11. Losses come from an equally weighted mixture of an exponential distribution with mean m_1, and an exponential distribution with mean m_2. Determine the least upper bound for the coefficient of variation of this distribution.

(A) 1 (B) $\sqrt{2}$ (C) $\sqrt{3}$ (D) 2 (E) $\sqrt{5}$

12. X has the following distribution : $Pr[X = 0] = 0.4$, $Pr[X = 1] = 0.6$.
The distribution of Y is conditional on the value of X: if $X = 0$, then the distribution of Y is $Pr[Y = 0] = 0.6$, $Pr[Y = 1] = 0.2$, $Pr[Y = 2] = 0.2$, and if $X = 1$, then the distribution of Y is $Pr[Y = 0] = 0.2$,$Pr[Y = 1] = 0.3$, $Pr[Y = 2] = 0.5$.
Z is the sum of Y independent normal random variables, each with mean and variance 2.

What is $Var[Z]$?

(A) 5.0 (B) 6.0 (C) 7.0 (D) 8.0 (E) 9.0

13. Subway trains arrive at your station at a Poisson rate of 20 per hour. 25% of the trains are express and 75% are local. The types and number of trains arriving are independent. An express gets you to work in 16 minutes and a local gets you there in 28 minutes. You always take the first train to arrive. Your co-worker always takes the first express. You are both waiting at the same station.

Calculate the conditional probability that you arrive at work before your co-worker, given that a local arrives first.

(A) 37% (B) 40% (C) 43% (D) 46% (E) 49%

Section 7 Problem Set Solutions

1. (a) Y is the mixture of two Poisson distributions.
$E[Y] = 0.8 \times E[X_L] + 0.2 \times E[X_H] = 0.8 \times 0.250 + 0.2 \times 2 = 0.6$.
To find the second moment of Y, we recall that for a Poisson distribution with mean λ, the variance is also λ. Since $Var[X] = E[X^2] - (E[X])^2$, it follows that $E[X^2] = Var[X] + (E[X])^2 = \lambda + \lambda^2$ for a Poisson distribution.

Then, $E[Y^2] = 0.8 \times E[X_L^2] + 0.2 \times E[X_H^2] = 0.8 \times (0.25 + 0.25^2) + 0.2 \times (2 + 2^2) = 1.45$.
Then $Var[Y] = 1.45 - (0.6)^2 = 1.09$.

(b) $$\begin{aligned} P(Y \leq 1) &= 0.8 \times P(X_1 \leq 1) + 0.2 \times P(X_2 \leq 1) \\ &= 0.8 \times [e^{-0.25} + 0.25e^{-0.25}] + 0.2 \times [e^{-2} + 2e^{-2}] = 0.860. \end{aligned}$$

2. The probability is a mixture of the probabilities for the two components.
$P(X \leq 200) = 0.8 \times P(X_1 \leq 200) + 0.2 \times P(X_2 \leq 200)$, where X_1 and X_2 are the two Pareto distributions.

$P(X_1 \leq 200) = 1 - \left(\frac{100}{200+100}\right)^2 = .8889$, and
$P(X_2 \leq 200) = 1 - \left(\frac{3000}{200+3000}\right)^4 = .2275$.
$P(X \leq 200) = (0.8 \times 0.8889) + (0.2 \times 0.2275) = 0.757$.

Answer A

3. $$\begin{aligned} P(N = 2) &= p \times P(N_1 = 2) + (1 - p) \times P(N_2 = 2) \\ &= p \times (0.5)^2 + (1 - p) \times 6 \times (0.5)^4 = 0.375 - 0.125p. \end{aligned}$$

We have used the binomial probabilities of the form $\binom{m}{k} q^k (1 - q)^{m-k}$.

Answer E

4. $f_Y(y) = \frac{1}{2} \times e^{-y} + \frac{1}{2} \times \frac{1}{3} e^{-y/3}$. $F_Y(y) = \frac{1}{2} \times (1 - e^{-y}) + \frac{1}{2} \times (1 - e^{-y/3})$.
$F_Z(z) = F_Y(\frac{z}{2}) = \frac{1}{2} \times (1 - e^{-z/2}) + \frac{1}{2} \times (1 - e^{-z/6})$.
Z is a mixture of two exponentials with means 2 and 6.

Answer D

5. $$f_Y(y) = 0.5 \times f_{X_1}(y) + 0.5 \times f_{X_2}(y)$$
$$= \begin{cases} 0.5 \times 0.001 + 0.5 \times 0.0005 = 0.00075 & \text{if } 0 < y < 1000 \\ 0.5 \times 0.0005 = 0.00025 & \text{if } 1000 \leq y < 2000 \end{cases}$$
$$F_Y(y) = 0.5 \times F_{X_1}(y) + 0.5 \times F_{X_2}(y)$$
$$= \begin{cases} 0.5 \times 0.001x + 0.5 \times 0.0005x = 0.00075x & \text{if } 0 < y < 1000 \\ 0.5 \times 1 + 0.5 \times 0.0005x = 0.5 + 0.00025x & \text{if } 1000 \leq y < 2000 \end{cases}$$

The median of Y, m, must satisfy the equation $F_Y(m) = 0.5$.
We see that at $y = 1000$, $F_Y(1000) = 0.75$. Therefore, $m < 1000$, so that $F_Y(m) = 0.00075m = 0.5$. Therefore, $m = 666.67$.
Note that the mean of Y is $0.5 \times 500 + 0.5 \times 1000 = 750$.

6. For a randomly chosen individual at age 50, the mortality probability is the mixture of the mortality probabilities for non-smokers and smokers. This is $q = \frac{2}{3} \times 0.1 + \frac{1}{3} \times 0.2 = \frac{2}{15}$.

   The probability that both of two independent 50-year old individuals survive the year is $(1 - \frac{2}{15})^2 = 0.7511$. The probability at least one of them dies by age 51 is $1 - 0.7511 = 0.25$.

   Suppose that there are 1000 non-smokers and 500 smokers at age 50.
   The expected number of surviving non-smokers at age 51 is $1000 \times 0.9 = 900$, and the number of surviving smokers is $500 \times 0.8 = 400$.

   A randomly chosen survivor at age 51 has a $\frac{9}{13}$ chance of being a non-smoker and a $\frac{4}{13}$ chance of being a smoker.

   The mortality probability at age 51 for the randomly chosen survivor is
   $\frac{9}{13} \times 0.1 + \frac{4}{13} \times 0.2 = .131$.

7. $f_Y(y) = \frac{2}{3} \times e^{-y} + \frac{1}{3} \times \frac{1}{2} e^{-y/2}$.

   $F_Y(y) = \frac{2}{3} \times (1 - e^{-y}) + \frac{1}{3} \times (1 - e^{-y/2})$.

   $E[Y] = \frac{2}{3} \times 1 + \frac{1}{3} \times 2 = \frac{4}{3}$.

   $E[Y^2] = \frac{2}{3} \times (2 \times 1^2) + \frac{1}{3} \times (2 \times 2^2) = 4 \rightarrow Var[Y] = 4 - (\frac{4}{3})^2 = \frac{20}{9}$.

   The 90th percentile of Y is c, which must satisfy the equation
   $F_Y(c) = \frac{2}{3} \times (1 - e^{-c}) + \frac{1}{3} \times (1 - e^{-c/2}) = 0.9$.

   If we let $r = e^{-c/2}$, then this becomes a quadratic equation in r,
   $\frac{2}{3} \times (1 - r^2) + \frac{1}{3} \times (1 - r) = 0.9$, or equivalently, $2r^2 + r - .3 = 0$.

   Solving for r results in $r = .21$ or $r = -0.71$.

   We ignore the negative root, since $r = e^{-c/2}$ must be positive.

   Then $c = -2 \ln r = -2 \ln .21 = 3.12$ is the 90th percentile of Y.

8. I. If $f_Y(y) = a_1 \times f_{X_1}(y) + a_2 \times f_{X_2}(y) = a_1 \times \frac{1}{\theta_1} e^{-y/\theta_1} + a_2 \times \frac{1}{\theta_2} e^{-y/\theta_2}$,
   and $a_1\theta_1 + a_2\theta_2 = 2$. Then $\theta_1 < 2 < \theta_2$ (or vice-versa).

   For an exponential distribution with mean 2, we have $f_Z(y) = \frac{1}{2} e^{-y/2}$
   $\frac{f_Y(y)}{f_Z(y)} = \frac{a_1 \times \frac{1}{\theta_1} e^{-y/\theta_1} + a_2 \times \frac{1}{\theta_2} e^{-y/\theta_2}}{\frac{1}{2} e^{-y/2}} = \frac{2a_1}{\theta_1} e^{y(\frac{1}{2} - \frac{1}{\theta_1})} + \frac{2a_2}{\theta_2} e^{y(\frac{1}{2} - \frac{1}{\theta_2})}$.

   Since $\theta_1 < 2 < \theta_2$, it follows that $\frac{1}{\theta_2} < \frac{1}{2} < \frac{1}{\theta_1}$. Therefore, $\frac{1}{2} - \frac{1}{\theta_2} > 0$
   so that $\lim_{y\to\infty} \frac{f_Y(y)}{f_Z(y)} = \lim_{y\to\infty} \frac{2a_1}{\theta_1} e^{y(\frac{1}{2} - \frac{1}{\theta_1})} + \frac{2a_2}{\theta_2} e^{y(\frac{1}{2} - \frac{1}{\theta_2})} = 0 + \infty$. True.

   II. Suppose that $f_Y(y) = \frac{1}{2} e^{-y} + \frac{1}{6} e^{-y/3}$. Then $S_Y(y) = \frac{1}{2} e^{-y} + \frac{1}{2} e^{-y/3}$,
   so that Y is mixture of exponentials with means of 1 and 3, with equal mixing weights of $\frac{1}{2}$.
   $e_Y(d) = \frac{\int_d^\infty S(y)\,dy}{S(d)} = \frac{\frac{1}{2} e^{-d} + \frac{3}{2} e^{-d/3}}{\frac{1}{2} e^{-d} + \frac{1}{2} e^{-d/3}}$.
   $e_{X_1}(d) = 1$, $e_{X_3}(d) = 3 \rightarrow \frac{1}{2} \times e_{X_1}(d) + \frac{1}{2} \times e_{X_3}(d) = 2$. False.

9. Since the two estimates are independent,

$$Var[\mu_C] = Var[w \times \mu_A + (1-w) \times \mu_B] = w^2 \times Var[\mu_A] + (1-w)^2 \times Var[\mu_B]$$
$$= w^2 \times 400^2 + (1-w)^2 \times 200^2 = 200{,}000w^2 - 80{,}000w + 40{,}000.$$

This is a quadratic expression in w with positive coefficient of w^2. The minimum can be found by differentiating with respect to w and setting equal to 0:
$400{,}000w - 80{,}000 = 0 \;\rightarrow\; w = \frac{1}{5}$ is the value of w that minimizes $Var[\mu_c]$.

Answer B

10. If Y is a mixture of X_1 and X_2 with mixing weights a and $1-a$, we can define the parameter $\Theta = \{1,2\}$, with $P[\Theta = 1] = a$, $P[\Theta = 2] = 1 - a$.
Then $E[Y] = E[\,E[Y|\Theta]\,] = E[Y|\Theta = 1] \times P[\Theta = 1] + E[Y|\Theta = 2] \times P[\Theta = 2]$
$= E[X_1] \times a + E[X_2] \times (1-a)$
(this is the usual way the mean of a finite mixture is formulated).

$Var[Y] = E[\,Var[Y|\Theta]\,] + Var[\,E[Y|\Theta]\,]$, where
$E[\,Var[Y|\Theta]\,] = Var[X_1] \times a + Var[X_2] \times (1-a)$ and
$Var[\,E[Y|\Theta]\,] = (E[X_1] - E[X_2])^2 \times a \times (1-a)$.

The last equality follows from the fact that if Z is a two-point random variable
$Z = \begin{cases} u & \text{prob.}\, p \\ v & \text{prob.}\, 1-p \end{cases}$, then $Var[Z] = (u-v)^2 \times p \times (1-p)$; $E[Y|\Theta]$ is
a two-point random variable $E[Y|\Theta] = \begin{cases} E[X_1] & \text{prob.}\, a \\ E[X_2] & \text{prob.}\, 1-a \end{cases}$.
Therefore $Var[Y] = Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a \times (1-a)$.

B) If X_1,X_2 are Poisson random variables then $E[X_1] = Var[X_1]$ and $E[X_2] = Var[X_2]$, so that $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) = Var[X_1] \times a + Var[X_2] \times (1-a)$.

It follows that
$Var[Y] = Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a \times (1-a) > E[Y]$.

C) For a negative binomial random variable with parameters r and β, the mean is $r\beta$ and the variance is $r\beta(1+\beta)$, so the variance is larger than the mean. If X_1 and X_2 have negative binomial distributions, the $E[X_1] < Var[X_1]$ and $E[X_2] < Var[X_2]$.

Therefore, $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) < Var[X_1] \times a + Var[X_2] \times (1-a)$, and
$Var[X_1] \times a + Var[X_2] \times (1-a)$
$< Var[X_1] \times a + Var[X_2] \times (1-a) + (E[X_1] - E[X_2])^2 \times a(1-a) = Var[Y]$;
therefore $E[Y] < Var[Y]$.

A) For a binomial random variable with parameters m and q, the mean is mq and the variance is $mq(1-q)$, which is smaller than the mean.

Therefore $E[Y] = E[X_1] \times a + E[X_2] \times (1-a) > Var[X_1] \times a + Var[X_2] \times (1-a)$, and it is possible that when we add $(E[X_1] - E[X_2])^2 \times a \times (1-a)$ to the right side, we get approximate equality.
So it is possible that $E[Y] = Var[Y]$ for a mixture of binomials.

Answer A

11. The mean will be $\frac{1}{2} \times (m_1 + m_2)$ and the variance will be $\frac{1}{2} \times (2m_1^2 + 2m_2^2) - [\frac{1}{2} \times (m_1 + m_2)]^2$. The coefficient of variation is the ratio of standard deviation to mean.
The square of the coefficient of variation is

$$\frac{\frac{1}{2} \times (2m_1^2 + 2m_2^2) - [\frac{1}{2} \times (m_1 + m_2)]^2}{[\frac{1}{2} \times (m_1 + m_2)]^2} = \frac{3m_1^2 - 2m_1m_2 + 3m_2^2}{(m_1 + m_2)^2}$$
$$= \frac{3m_1^2 + 6m_1m_2 + 3m_2^2 - 8m_1m_2}{(m_1 + m_2)^2}$$
$$= 3 - \frac{8m_1m_2}{(m_1 + m_2)^2}.$$

The maximum square of the coefficient of variation is 3, and it occurs at the minimum of $\frac{8m_1m_2}{(m_1+m_2)^2}$, which is 0 (if m_1 or m_2 is 0).
The least upper bound of the coefficient of variation is $\sqrt{3}$.

Answer C

12. $Var[Z] = Var[\,E[Z \mid Y]\,] + E[\,Var[Z \mid Y]\,]$.

$E[Z \mid Y] = 2 \times Y$ (a sum of Y independent normal random variables each with mean 2), and $Var[Z \mid Y] = 2 \times Y$ (a sum of Y independent normal random variables each with a variance of 2).

Thus, $Var[Z] = Var[2 \times Y] + E[2 \times Y] = 4 \times Var[Y] + 2 \times E[Y]$.
$Var[Y] = Var[\,E[Y \mid X]\,] + E[\,Var[Y \mid X]\,]$.
Since $E[Y] = 0.6$ with prob. $0.4\,(X = 0)$ and $E[Y] = 1.3$ with prob. $0.6\,(X = 1)$, it follows that $Var[E[Y \mid X]] = (0.6)^2 \times 0.4 + (1.3)^2 \times 0.6 - (1.02)^2 = 0.1176$.

Also, if $X = 0$, the variance of Y is $1^2 \times 0.2 + 2^2 \times 0.2 - (0.6)^2 = 0.64$,
and if $X = 1$, the variance of Y is $1^2 \times 0.3 + 2^2 \times 0.5 - (1.3)^2 = 0.61$. Thus,
$E[Var[Y \mid X]] = 0.64 \times 0.4 + .61 \times .6 = 0.622$, and so $Var[Y] = 0.1176 + 0.622 = 0.7396$.

Then, the variance of Z is $4 \times 0.7396 + 2 \times 1.02 = 4.9984$. Note that $Var[Y]$ can also be found from $E[Y^2] - (E[Y])^2$. Again, $E[Y] = 1.02$, as above, and now,
$E[Y^2] = E[E[Y^2 \mid X]] = 1 \times 0.4 + 2.3 \times 0.6 = 1.78$, so that $Var[Y] = 0.7396$.

Answer A

13. Given that a local train arrives first, you will get to work 28 minutes after that local train arrives, since you will take it. Your co-worker will wait for first express train. You will get to work before your co-worker if the next express train (after the local) arrives more than 12 minutes after the local. We expect 5 express trains per hour, so the time between express trains is exponentially distributed with a mean of $\frac{1}{5}$ of an hour, or 12 minutes. Because of the lack of memory property of the exponential distribution, since we are given that the next train is local, the time until the next express train after that is exponential with a mean of 12 minutes. Therefore, the probability that after the local, the next express arrives in more than 12 minutes is $P[T > 12]$, where T has an exponential distribution with a mean of 12. This probability is $e^{-12/12} = e^{-1} = 0.368$ (37%).

Answer A

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Practice Exam 1

1. The XYZ Insurance Company sells property insurance policies with a deductible of \$5,000, policy limit of \$500,000, and a coinsurance factor of 80%. Let X_i be the individual loss amount of the ith claim and Y_i be the claim payment of the ith claim. Which of the following represents the relationship between X_i and Y_i?

(A) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 625{,}000 \\ 500{,}000 & X_i > 625{,}000 \end{cases}$

(B) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 4{,}000) & 4{,}000 < X_i \leq 500{,}000 \\ 500{,}000 & X_i > 500{,}000 \end{cases}$

(C) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 630{,}000 \\ 500{,}000 & X_i > 630{,}000 \end{cases}$

(D) $Y_i = \begin{cases} 0 & X_i \leq 6{,}250 \\ 0.80\,(X_i - 6{,}250) & 6{,}250 < X_i \leq 631{,}500 \\ 500{,}000 & X_i > 631{,}500 \end{cases}$

(E) $Y_i = \begin{cases} 0 & X_i \leq 5{,}000 \\ 0.80\,(X_i - 5{,}000) & 5{,}000 < X_i \leq 505{,}000 \\ 500{,}000 & X_i > 505{,}000 \end{cases}$

2. A casino has a game that makes payouts at a Poisson rate of 5 per hour and the payout amounts can be any non-negative integer, 1, 2, 3, ... without limit. The probability that any given payout is equal to $i > 0$ is $\frac{1}{2^i}$. Payouts are independent. Calculate the probability that there are no payouts of 1,2, or 3 in a given 20 minute period.

(A) 0.08 (B) 0.13 (C) 0.18 (D) 0.23 (E) 0.28

3. A compound Poisson claim distribution has $\lambda = 3$ and individual claims amounts distributed as follows:

x	$f_X(x)$
5	0.6
10	0.4

Determine the expected cost of an aggregate stop-loss insurance with a deductible of 6.

(A) Less than 15.0

(B) At least 15.0 but less than 15.3

(C) At least 15.3 but less than 15.6

(D) At least 15.6 but less than 15.9

(E) At least 15.9

Use the following information for questions 4 and 5. You are the producer of a television quiz show that gives cash prizes. The number of prizes, N, and prize amounts, X are independent of one another and have the following distributions:

N: $P[N = 1] = 0.8$, $P[N = 2] = 0.2$

X: $P[X = 0] = 0.2$, $P[X = 100] = 0.7$, $P[X = 1000] = 0.1$

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4. Your budget for prizes equals the expected prizes plus $1.25\times$ standard deviation of prizes. Calculate your budget.

(A) 384 (B) 394 (C) 494 (D) 588 (E) 596

5. You buy stop-loss insurance for prizes with a deductible of 200. The cost of insurance includes a 175% relative security load (the relative security load is the percentage of expected payment that is added). Calculate the cost of the insurance.

(A) 204 (B) 227 (C) 245 (D) 273 (E) 357

6. An actuary determines that claim counts follow a negative binomial distribution with unknown β and r. It is also determined that individual claim amounts are independent and identically distributed with mean 700 and variance 1,300. Aggregate losses have a mean of 48,000 and variance 80 million. Calculate the values for β and r.

(A) $\beta = 1.20, r = 57.19$

(B) $\beta = 1.38, r = 49.75$

(C) $\beta = 2.38, r = 28.83$

(D) $\beta = 1{,}663.81, r = 0.04$

(E) $\beta = 1{,}664.81, r = 0.04$

7. Let X_1, X_2, X_3 be independent Poisson random variables with means $\theta, 2\theta$, and 3θ respectively. What is the maximum likelihood estimator of θ based on sample values x_1, x_2, and x_3 from the distributions of X_1, X_2 and X_3, respectively,

(A) $\dfrac{\bar{x}}{2}$

(B) $\bar{x}$

(C) $\dfrac{x_1 + 2x_2 + 3x_3}{6}$

(D) $\dfrac{3x_1 + 2x_2 + x_3}{6}$

(E) $\dfrac{6x_1 + 3x_2 + 2x_3}{11}$

8. For a group of policies, you are given:

(i) Losses follow a uniform distribution on the interval $(0, \theta)$, where $\theta > 25$.

(ii) A sample of 20 losses resulted in the following:

Interval	Number of Losses
$x \leq 10$	n_1
$10 < x \leq 25$	n_2
$x > 25$	n_3

The maximum likelihood estimate of θ can be written in the form $25 + y$. Determine y.

(A) $\dfrac{25n_1}{n_2 + n_3}$

(B) $\dfrac{25n_2}{n_1 + n_3}$

(C) $\dfrac{25n_3}{n_1 + n_2}$

(D) $\dfrac{25n_1}{n_1 + n_2 + n_3}$

(E) $\dfrac{25n_2}{n_1 + n_2 + n_3}$

9. The number of claims follows a negative binomial distribution with parameters β and r, where β is unknown and r is known. You wish to estimate β based on n observations, where $\bar{x}$ is the mean of these observations. Determine the maximum likelihood estimate of β.

(A) $\dfrac{\bar{x}}{r^2}$ (B) $\dfrac{\bar{x}}{r}$ (C) $\bar{x}$ (D) $r\bar{x}$ (E) $r^2\bar{x}$

10. The following 6 observations are assumed to come from the continuous distribution with pdf $f(x;\theta) = \frac{1}{2}x^2\theta^3e^{-\theta x}$: 1, 3, 4, 4, 5, 7.

Find the mle of θ.

(A) 0.25 (B) 0.50 (C) 0.75 (D) 1.00 (E) 1.25

11. An analysis of credibility premiums is being done for a particular compound Poisson claims distribution, where the criterion is that the total cost of claims is within 5% of the expected cost of claims with a probability of 90%. It is found that with $n = 60$ exposures (periods) and $\bar{X} = 180.0$, the credibility premium is 189.47. After 20 more exposures (for a total of 80) and revised $\bar{X} = 185$, the credibility premium is 190.88. After 20 more exposures (for a total of 100) the revised $\bar{X}$ is 187.5. Assuming that the manual premium remains unchanged in all cases, and assuming that full credibility has not been reached in any of the cases, find the credibility premium for the 100 exposure case.

(A) 191.5 (B) 192.5 (C) 193.5 (D) 194.5 (E) 196.5

12. For an insurance portfolio, you are given:

(i) For each individual insured, the number of claims follows a Poisson distribution.

(ii) The mean claim count varies by insured, and the distribution of mean claim counts follows gamma distribution.

(iii) For a random sample of 1000 insureds, the observed claim counts are as follows:

Number of Claims, n	0	1	2	3	4	5
Number of Insureds, f_n	512	307	123	41	11	6

$$\sum nf_n = 750, \qquad \sum n^2 f_n = 1494$$

(iv) Claim sizes follow a Pareto distribution with mean 1500 and variance 6,750,000.

(v) Claim sizes and claim counts are independent.

(vi) The full credibility standard is to be within 5% of the expected aggregate loss 95% of the time.

Determine the minimum number of insureds needed for the aggregate loss to be fully credible.

(A) Less than 8300

(B) At least 8300, but less than 8400

(C) At least 8400, but less than 8500

(D) At least 8500, but less than 8600

(E) At least 8600

Information on Questions 13 and 14 is as follows. You are given the following information on cumulative incurred losses through development years shown.

	Cumulative Incurred Losses				Paid-to-Date
Accident	Development Year				at Dec 31, AY4
Year	0	1	2	3	
AY1	2325	3749	4577	4701	4701
AY2	2657	4438	5529		4500
AY3	2913	4995			3500
AY4	3163				2500

13. Using an average factor model, calculate the estimated total loss reserve as of Dec. 31, AY4.

(A) Less than 7500

(B) At least 7500 but less than 7800

(C) At least 7800 but less than 8100

(D) At least 8100 but less than 8400

(E) At least 8400

14. As of Dec. 31, AY4, calculate

Estimated reserve for AY2 based on an average factor model)
− (Estimated reserve for AY2 based on a mean factor model)

(A) Less than −300

(B) At least −300 but less than −100

(C) At least −100 but less than 100

(D) At least 100 but less than 300

(E) At least 300

15. For a one-period binomial model for the price of a stock with price 100 at time 0, you are given:

(i) The stock pays no dividends.

(ii) The stock price is either 110 or 95 at the end of the year.

(iii) The risk free force of interest is 5%.

Calculate the price at time 0 of a one-year call option with strike price 100.

(A) Less than 6.00

(B) At least 6.00 but less than 6.25

(C) At least 6.25 but less than 6.50

(D) At least 6.50 but less than 6.75

(E) At least 6.75

16. Using the following information, determine the incurred losses for the 2017 accident year as reported at Dec. 31, 2018.

Occurrence #1: Occurrence date Feb. 1/16, Report date Apr. 1/16

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Apr. 1/16	1000	1000	2000
Dec. 31/16	1500	1000	2500
Dec. 31/17	1500	1000	2500
Mar. 1/18	3000	0	3000

Occurrence #2: Occurrence date May 1/17, Report date July 1/17

Loss History:

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
July 1/17	1000	2000	3000
Dec. 31/17	3000	1000	4000
Dec. 31/18	5000	0	5000

Occurrence #3: Occurrence date Nov. 1/17, Report date Feb. 1/18

Loss History

Date	Total Paid to Date	Unpaid Loss Reserve	Total Incurred
Mar. 1/18	0	8000	8000
Dec. 31/18	5000	5000	10,000

(A) 0 (B) 3,000 (C) 5,000 (D) 8,000 (E) 15,000

17. You are given the following calendar year earned premium.

Year	CY2	CY3	CY4
Earned Premium	4200	4700	5000

You are also given the following rate changes

Date	April 1, CY1	September 1, CY2	July 1, CY3
Average Rate Change	+12 %	+6 %	+10 %

Determine the approximate earned premium at current (end of CY4) rates for CY3.

(A) Less than 5000

(B) At least 5000 but less than 5100

(C) At least 5100 but less than 5200

(D) At least 5200 but less than 5300

(E) At least 5300

**** END OF EXAMINATION ****

Solutions to Practice Exam 1

1.	C	
2.	D	
3.	C	
4.	E	
5.	D	
6.	B	
7.	A	
8.	C	
9.	B	
10.	C	
11.	A	
12.	E	
13.	D	
14.	C	
15.	C	
16.	E	
17.	C	

1. With coinsurance factor α, deductible d, policy limit $\alpha(u-d)$, the amount paid per loss is (we are assuming in inflation rate of $r=0$) $Y=\begin{cases} 0 & X\le d \\ \alpha(X-d) & d<X\le u. \\ \alpha(u-d) & X>u \end{cases}$

   In this problem, the coinsurance factor is $\alpha=.8$, the deductible is $d=5{,}000$, and the policy limit is $.8(u-5{,}000)=500{,}000$, so that the maximum covered loss is $u=630{,}000$.

   The amount paid per loss becomes $Y=\begin{cases} 0 & X\le 5{,}000 \\ 0.80(X-5{,}000) & 5{,}000<X\le 630{,}000 \\ 500{,}000 & X>630{,}000 \end{cases}$

   Answer C

2. When a payout occurs, it is 1, 2 or 3 with probability $\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}=\frac{7}{8}$. The number of payouts that are 1,2 or 3 follows a Poisson process with an hourly rate of $5\times\frac{7}{8}=\frac{35}{8}$.
   The expected number of payouts that are 1, 2 or 3 in 20 minutes, say N, has a Poisson distribution with mean $\frac{35}{8}\times\frac{20}{60}=\frac{35}{24}$. The probability that there are no payouts of 1, 2, or 3 in a given 20 minute period is the probability that $N=0$, which is $e^{-35/24}=.233$.

   Answer D

3. The minimum claim amount is 5 if a claim occurs. S must be 0 or a multiple of 5 .
   The stop-loss insurance with deductible 6 pays $(S-6)_+=S-(S\wedge 6)$,

   where $S\wedge 6=\begin{cases} 0 & S=0 \\ 5 & S=5 \\ 6 & S\ge 10 \end{cases}$

   $E[S]=E[N]\times E[X]=(3)[(5)(.6)+(10)(.4)]=21$
   $E[S\wedge 6]=5\times P(S=5)+6[1-P(S=0{,}5)]$.
   $P(S=0)=P(N=0)=e^{-3}$ and
   $P(S=5)=P(N=1)\times P(X=5)=e^{-3}\times 3\times(.6)=1.8e^{-3}$.
   $E[S\wedge 6]=5\left(1.8e^{-3}\right)+6[1-P(S=0{,}5)]=6\left[1-2.8e^{-3}\right]=5.61$.
   Then $E\left[(S-6)_+\right]=E[S]-E[S\wedge 6]=21-5.61=15.39$.

   Answer C

4. This problem involves a compound distribution.
   The frequency (number of prizes) is N and the severity (prize amount) is X.
   The aggregate prize amount is
   $S=X_1+X_2+\times s+X_N$, with mean $E[S]=E[N]\times E[X]=(1.2)(170)=204$ and variance $Var[S]=E[N]\times Var[X]+Var[N]\times(E[X])^2$

   In this case, $Var[N]=E\left[N^2\right]-(E[N])^2=1.6-(1.2)^2=.16$, and
   $Var[X]=E\left[X^2\right]-(E[X])^2=107{,}000-(170)^2=78{,}100$

   Then, $Var[S]=(1.2)(78{,}100)+(.16)(170)^2=98{,}344$.
   The budget is $E[S]+1.25\sqrt{Var[S]}=204+1.25\sqrt{98{,}344}=596$.

   Answer E