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Johnny Li, Ph.D., FSA

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QUESTION 19 OF 704 Question # Go! Prev Next

Question Difficulty: Advanced

An airport purchases an insurance policy to offset costs associated with excessive amounts of snowfall. The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700.

The following table shows the probability function for the random variable X of annual (winter season) snowfall, in inches, at the airport.

Inches	[0,20)	[20,30)	[30,40)	[40,50)	[50,60)	[60,70)	[70,80)	[80,90)	[90,inf)
Probability	0.06	0.18	0.26	0.22	0.14	0.06	0.04	0.04	0.00

Calculate the standard deviation of the amount paid under the policy.

Possible Answers

A 134 ✓ 235 X 271 D 313 E 352

Help Me Start

Find the probabilities for the four possible payment amounts: 0, 300, 600, and 700.

Solution

With the amount of snowfall as X and the amount paid under the policy as Y, we have

y	$f_Y(y) = P(Y=y)$
0	$P(Y=0) = P(0 \le X < 50) = 0.72$
300	$P(Y=300) = P(50 \le X < 60) = 0.14$
600	$P(Y=600) = P(60 \le X < 70) = 0.06$
700	$P(Y=700) = P(Y \ge 70) = 0.08$

The standard deviation of Y is $\sqrt{E(Y^2) - [E(Y)]^2}$.

$$E(Y) = 0.14 \times 300 + 0.06 \times 600 + 0.08 \times 700 = 134$$

$$E(Y^2) = 0.14 \times 300^2 + 0.06 \times 600^2 + 0.08 \times 700^2 = 73400$$

$$\sqrt{E(Y^2) - [E(Y)]^2} = \sqrt{73400 - 134^2} = 235.465$$

Common Questions & Errors

Students shouldn't overthink the problem with fractional payments of 300. Also, account for probabilities in which payment cap of 700 is reached.

In these problems, we must distinguish between the REALT RV (how much snow falls) and the PAYMENT RV (when does the insurer pay)?. The problem states "The insurer pays the airport 300 for every full ten inches of snow in excess of 40 inches, up to a policy maximum of 700 ." So the insurer will not start paying UNTIL AFTER 10 full inches in excess of 40 inches of snow is reached (say at 50+ or 51). In other words, the insurer will pay nothing if X<50.

Rate this problem Excellent Needs Improvement Inadequate

Quickly access the Hub for additional learning.

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Helpful strategies to get you started.

Full solutions with detailed explanations to deepen your understanding.

Commonly encountered errors.

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Finance, Investments, & Derivatives

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Publisher's Note

The **Finance, Investments, and Derivatives** textbook by ACTEX is a continuation of the high-quality educational resources we have provided since 1972. This book is based on the ACTEX study manual for Exam IFM (Investments and Financial Markets), which was part of the Society of Actuaries (SOA) curriculum from 2018 until its final administration in November 2022.

Although Exam IFM is no longer part of the ASA pathway, the foundational topics it covered—spanning finance, investments, and derivatives—remain vital for actuarial practice and broader applications in financial and risk management fields. Consequently, this textbook serves as a comprehensive reference for students, professionals, and educators, providing insights into:

- Investment risk and return
- Portfolio theory and the Capital Asset Pricing Model (CAPM)
- Corporate finance, including capital structure and financing methods
- Financial derivatives, such as forwards, options, and futures
- Option valuation methods, including binomial trees and the Black–Scholes model
- Applications of option pricing in actuarial science and beyond

The layout of the textbook has been preserved, with detailed lessons, examples, and exercises designed to reinforce understanding and simulate exam-like scenarios. Complete solutions are included to provide readers with a clear path to mastering the material. In addition, you will find five mock tests to help you gauge your progress and readiness.

Our decision to publish this textbook reflects our commitment to supporting the continued learning and professional development of actuarial students and professionals. By retaining the content and structure of the original study manual, this resource ensures that the knowledge gained through studying Exam IFM remains accessible and relevant—even as the actuarial curriculum evolves.

We hope this textbook serves as a reliable companion for readers navigating the complex and dynamic fields of finance, investments, and derivatives. Thank you for choosing ACTEX as your trusted partner in education.

Yijia Liu

Director of Editorial Management

ACTEX Learning

December, 2024

Preface

Thank you for choosing ACTEX Learning.

The Investment and Financial Markets (IFM) Exam is a new exam that is first launched by the Society of Actuaries (SoA) in July 2018. Although the IFM Exam draws heavily from the MFE Exam (which is no longer offered after July 1, 2018), it covers a lot of topics (including corporate finance and the interface between derivatives and insurance) which the MFE Exam does not cover. This brand new study manual is created to help you best prepare for the IFM Exam.

Given that the IFM Exam covers a very wide range of topics, it is crucial to learn them in a logical sequence and to see through the connections among them. We have meticulously categorized the exam topics into two broad themes: Quantitative and Qualitative.

The first part of this manual focuses on the Quantitative theme, which encompasses all of the topics covered in *Derivatives Markets* (the required text authored by R.L. McDonald) and the technical topics from *Corporate Finance* (the required text authored by J. Berk and others). To help you develop a strong foundation, we begin with the easiest calculations that are just straightforward extensions of what you have learnt in Exam FM. These are then followed by progressively harder calculations, ranging from the binomial model to various versions of the Black-Scholes formula.

The following features concerning the Quantitative part of the manual are noteworthy:

1. The connections between the option pricing models (from *Derivatives Markets*) and real options (from *Corporate Finance*) are clearly explained.

2. We do not want to overwhelm readers with verbose explanations. Whenever possible, concepts and techniques are demonstrated with examples and/or integrated into the practice problems.

3. We provide sufficient practice problems (which are similar to the real exam problems in terms of format and level of difficulty), so that you do not have to go through the textbooks' end-of-chapter problems. We find that the end-of-chapter problems in *Derivatives Markets* are either too trivial (simple substitutions) or too computationally intensive (Excel may be required), compared to the real exam questions.

4. We do not follow the order in Derivatives Markets, because the focus of this textbook is somewhat different from what the SoA expects from the candidates. According to the SoA, the purpose of the exam is "to develop candidates' knowledge of the theoretical basis," but the book emphasizes more on applications.

We believe that the materials in the Quantitative theme should be studied first, as a lot of time has to be spent on the practice problems in order to develop a solid mastery of these materials.

The second part of the manual is devoted to the Qualitative theme, which encompasses a lot of definitions and hard facts that you have to memorize (unfortunately). There are some calculations in the Qualitative theme, but they are typically trivial. To help you breeze through this theme, the materials in this theme are presented in an easy-to-read point form, with the most important points being clearly highlighted. Of course, we have practice problems to test how well you can remember the materials.

We recommend that you go through the Qualitative theme after the Quantitative theme, simply because everyone's short-term memory is limited.

The manual is concluded with several mock exams, which are written in a similar format to the released exam and sample questions provided by the SoA. This will enable you to, for example, retrieve information more quickly in the real exam. Further, we have integrated all of the relevant released exam and sample questions into the examples, practice problems, mock exams in the manual. These exam/sample questions include:

– The released MFE sample and released exam questions that are still relevant to the IFM exam syllabus.

– The released FM sample and released exam questions that are relevant to the IFM exam syllabus.

– The sample questions on Finance and Investment (Corporate Finance, IFM-21-18, IFM-22-18).

This integration seems to be a better way to learn how to solve those questions, and of course, you will need no extra time to review those questions.

We recommend you to use of this study manual is as follows:

1. Read the lessons in order.
2. Immediately after reading a lesson, complete the practice problems for that lesson.
3. After studying all lessons, work on the mock exams.

If you find a possible error in this manual, please let us know at the "Customer Feedback" link on the ACTEX homepage (www.actexmadriver.com). Any confirmed errata will be posted on the ACTEX website under the "Errata & Updates" link.

A Note on Rounding and the Normal Distribution Calculator

To achieve the desired accuracy, we recommend that you store values in intermediate steps in your calculator. If you prefer not to, please keep at least six decimal places.

In this study guide, normal probability values and z-values are based on a normal distribution calculator instead of a normal table in other exams. In the actual examination you will be able to use the same normal distribution calculator.

The calculator is very easy to use. Simply go to

https://www.prometric.com/en-us/clients/SOA/Pages/calculator.aspx

Recall that $N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} dz$ is the cumulative distribution function of a standard normal random variable. To find $N(x)$, you may use the first panel of the calculator. Type in the value of x and press "Normal CDF". Then you would get $N(x)$. For example, when $x = -1.282$, the calculator would report 0.09992.

To find the 100pth percentile of the standard normal random variable (i.e. to find the value of x such that $N(x) = p$), enter p into the cell adjacent to $N(x)$, and press "Inverse CDF". Then you would get x. For example, when $N(x) = 0.25$, the calculator would report -0.67449.

If you do not want to go online every time when you follow the examples and work on the practice problems, you can set up your own normal distribution calculator using Excel. Open a blank workbook, and set up the following:

Cell A1: x

Cell A2: N(x)

Cell B1: −1.282

Cell B2: = round(norm.s.dist(B1, 1), 5)

Cell A5: N(x)

Cell A6: x

Cell B5: 0.25

Cell B6: = round(norm.s.inv(B5), 5)

Cell B2 would report 0.09992 and Cell B6 would report -0.67449 if you are using Excel 2010 or more recent versions of Excel. You can alter the values in B1 and B5 to calculate any probability and percentile. Save your workbook for later use.

Syllabus Reference

In what follows,

- BM stands for the textbook *Corporate Finance* (4th Ed) coauthored by Berk and DeMarzo,
- McD stands for the textbook *Derivatives Markets* (3rd Ed) authored by McDonald,
- SN1 stands for the SoA published Study Note IFM-21-18 *Measures of Investment Risk, Monte Carlo Simulation and Empirical Evidence on the Efficient Market Hypothesis*, and
- SN2 stands for the SoA published Study Note IFM-22-18 *Actuarial Applications of Options and Other Financial Derivatives.*

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1.1.1	BM 10.3, 10.4
1.1.2	BM 10.2, 11.2 (starting from the middle of p.360)
1.1.3	BM 11.1, 11.2, 11.3 (up to p.365, and start again at p.369)
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4.3.2	McD 10.5 (p.312 and 313), McD 9.1 (formula 9.4 only)
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5.4.2	McD 13.4 (p.394 to p.395)
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6.3.1	McD 9.3 (p.281 – 285 "Different strike prices")
6.3.2	McD 9.3 (p.276 – 277 "European versus American options" and "maximum and minimum option prices")
6.3.3	McD 9.3 (p.280 - 281 "Time to expiration")
6.3.4	McD 9.3 (from the middle of p.277 to the middle of p.278), 11.1
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Module 2

Project Analysis and Investment Risk

Lesson 1

Cost of Capital

OBJECTIVES

1. To determine the equity cost of capital
2. To determine the debt cost of capital
3. To determine a project's cost of capital

Armed with the materials in Module 1, we are now in a position to link the materials in Exam FM to the quantitative part of Exam IFM.

When a firm is deciding whether to accept a project with **fixed cash flows**, they either

(1) compute the **NPV** of the project using a **risk-free rate of return**, and accept the project if the NPV is positive; or

(2) compute the **internal rate of return** of the project (assume that there is one and only one IRR), and accept the project if the IRR is higher than some appropriate discount rate.

Since the IRR approach suffers from many problems (e.g. multiple IRR, selection of mutually exclusive projects), the NPV method is more widely used in capital budgeting.

However, for most projects, cash flows are **random**. Suppose that the risk-free rate is 10%. Consider two projects, both of which need an initial investment of 160.

Project 1: You would receive a certain cash flow of \$200 after 1 year. The NPV is

$$200/1.1 - 160 = 181.82 - 160 = 21.82.$$

Project 2: You would receive \$100, \$200 or \$300. Each possible outcome has a probability of 1/3. The expected cash flow after 1 year is also \$200.

Which project would you prefer? Since rational investors are risk-adverse, they would prefer the first project. This means that the NPV of the second project should be less than 21.82. Equivalently speaking, it is wrong to use a risk-free rate to discount risky cash flows.

So, how should we value random cash flows? The two methods commonly used in practice are:

(1) To reduce the expected cash flows appropriately (through adjusting the probabilities of possible outcomes) and use the risk-free rate to discount them. This is the **derivative approach**, which is detailed in Modules 4 and 5.

(2) To adjust the interest rate appropriately using the CAPM so that the expected value is discounted more heavily. The adjusted interest rate is called the **risk-adjusted interest / discount rate** or the **cost of capital**.

This lesson explores (2).

2.1.1 The Equity Cost of Capital

A firm needs capital to operate. There are mainly two ways to raise capital.

(1) A firm can issue only **stocks** (equity). Such a firm may pay dividends to its stockholders, but the payment of dividends is not an obligation.

(2) A firm can issue **debts** to raise capital. In this case, the firm is obligated to pay interest (unless the debt is zero-coupon) and repay the principal upon maturity.

(Both ways to raise capital will be discussed in Module 9.)

To attract investors, the stocks and debts have to offer a reasonable expected rate of return. Such returns are called the **equity cost of capital** and the **debt cost of capital**. Since debt is relatively safer than equity, for a given firm, the debt cost of capital is less than the equity cost of capital.

To determine the appropriate cost of capital for a project of a firm, we often need to first determine the costs of capital of the firm's (and in many cases many similar firm's) equity and debt. In Module 1 Lesson 3 we estimated the **beta** of a firm by performing a linear regression of the excess return on the firm's stock against the excess return on the market portfolio. The beta obtained is also called the **equity beta** of the firm. The expected return obtained from the SML is the **equity cost of capital** of the firm.

If a firm issues only stocks, then the firm's asset and equity must be equal. This means that the expected return on the firm's asset must be the same as the equity cost of capital. If the firm has a new project that has very similar risk characteristic as the firm's current projects (represented by the firm's asset), then we can use the equity cost of capital as the project's cost of capital. Let us take Walmart as an example. Suppose that an analyst is estimating the value of new Walmart outlet. The Walmart outlet is largely a clone of the existing outlets and its risk should closely match the overall risk of the entire Walmart Corporation, which is basically a collection of the cloned chain stores. (We describe such a project as **scale-enhancing**, meaning that the project carries the same amount and type of risk as other projects that the firm is currently running.) Hence, the cost of capital for Walmart as a whole seems to be an appropriate discount rate for evaluating the profitability of the new outlet.

Example (2.1.1)

An all-equity firm has a beta of 1.2. To value projects of the firm, you are given:

(i) The market risk premium is 9.5%.

(ii) The risk-free interest rate is 5%.

(iii) The firm is considering the following three mutually exclusive projects:

Project	Initial cost	Expected cash flow(s)
A	100	130 at time 1
B	94	120 at time 1
C	55	15 at times 1, 2, ..., 9

(a) Calculate the IRR for all three projects. Which project has the largest IRR?

(b) Assuming that each project has a similar risk as the firm, determine the project that the firm should pick.

Solution

(a) For project A: $100 = 130/(1 + \text{IRR}_A) \Rightarrow \text{IRR}_A = 30\%$

For project B: $94 = 120/(1 + \text{IRR}_B) \Rightarrow \text{IRR}_B = 27.66\%$

For project C: $55 = 15 \times a_9$. Using a financial calculator, we find that $\text{IRR}_C = 23.06\%$. So Project A has the largest IRR.

(b) The equity cost of capital is $0.05 + 1.2 \times 0.095 = 16.4\%$. This is also the rate we use for discounting.

$\text{NPV(A)} = 130/1.164 - 100 = 11.68.$

$\text{NPV(B)} = 120/1.164 - 94 = 9.09.$

$\text{NPV(C)} = 15 \times \frac{1-\frac{1}{1.164^9}}{0.164} - 55 = 15 \times 4.5431 - 55 = 13.15.$

So the firm should pick Project C.

[END]

A firm can certainly consider a project that is not the same as the existing projects. This is rather like when the McDonald's first opened a new line called McCafé in 2001 in Chicago. It can also happen that a firm has multiple lines of businesses, each with different risk characteristics. In such situations, the calculation in Example (2.1.1) would not work: we cannot use the firm's cost of capital as the project's cost of capital.

2.1.2 The Debt Cost of Capital

If a firm uses both debt and equity for financing, then the firm's asset and equity would have different risks. The use of debt increases the chance of bankruptcy and makes the equity risker. To calculate the beta of the firm's asset, we would need to use the debt cost of capital of the firm. In this section we illustrate how the debt cost of capital can be estimated and how we can relate the equity beta and debt beta to the firm's asset beta.

Industry Beta of Debt

The risk of default is higher for debts issued by firms with poor credit ratings and for debts with longer times to maturity. Some studies publish the average **debt betas** by rating and maturity. These published values can be used with the CAPM to estimate the **debt cost of capital**.

Tuning down the Yield to Maturity (YTM)

The **YTM** of a bond is the **IRR** of it, assuming that all the promised payments would be made in full. Taking the possibility of default into consideration, the expected return on the bond is in general smaller than the bond's YTM. Consider a one-year bond. Let L be the expected loss rate (measured as the fraction of the face value) given default, and p be the default probability. Then

$$\text{expected return of the bond} = (1-p)\text{YTM} + p(\text{YTM}-L) = \text{YTM}-pL.$$

(This formula also holds for a multi-year bond if p, L and YTM do not vary with time.)

Example (2.1.2)

An A- rated 5-year bond has a yield to maturity of 2.5%. You estimate that there is a 1% chance that the bond would default per year, and the expected loss rate given default is 70%.

(a) Estimate the expected return on the bond.

(b) If the risk-free interest is 1% and the market risk premium is 8%, estimate the bond's beta.

Solution

(a) The expected return is $2.5\% - 0.01 \times 70\% = 1.8\%$.

(b) Solving the equation $1.8\% = 1\% + \beta_D \times 8\%$, we get $\beta_D = 0.1$.

[END]

The beta of the debt is typically small. Even for a CCC bond (a bond that has an annual default rate bigger than 75% in bad economy), the beta is no more than 0.5.

Unlevered Cost of Capital and Beta

A firm's **asset** is made up of its equity and debt. The asset is like a portfolio of equity and debt. If we let r_U be the firm's expected return on the asset or the asset cost of capital (yet another name is unlevered cost of capital of the firm), then

$$r_U = \frac{E}{E+D} r_E + \frac{D}{E+D} r_D,$$

where r_E (r_D) is the equity (debt) cost of capital. Also, the firm's asset is sensitive to the market portfolio because its equity and debt are sensitive to the market portfolio. If we let β_U be the asset beta of a firm (or the unlevered beta), then

$$\beta_U = \frac{E}{E+D} \beta_E + \frac{D}{E+D} \beta_D,$$

where β_E (β_D) is the equity (debt) beta of the firm. The above two formulas work as long as $\frac{D}{E+D}$ is constant through time.

Sometimes a firm maintains a large cash balance. The cash can be treated as a risk-free asset that counter-weights the debt and lowers the firm's risk. We can calculate the firm's net debt by

$$\text{Net debt} = \text{Debt} - \text{Excess cash and short-term investments} = D - C.$$

In extreme cases, especially for high-tech firms that have very little debt and lots of cash for R&D, the net debt can be negative. The firm's **enterprise value** is the firm's equity plus debt, minus the excess cash, and its beta can be computed from

$$\beta_U = \frac{E}{E+D-C}\beta_E + \frac{D}{E+D-C}\beta_D - \frac{C}{E+D-C}\beta_C,$$

where β_C is the beta of the cash balance. In general, the cash balance is nearly risk-free and β_C is close to 0. The textbook mentions that if we assume that the cash balance and the firm's debt have similar market risk, or if the debt beta reflects the combined risk of the firm's debt and cash balance, then $\beta_D \approx \beta_C$, and we have the following shortcut:

$$\beta_U = \frac{E}{E+D-C}\beta_E + \frac{D-C}{E+D-C}\beta_D.$$

Example (2.1.3)

A high-tech firm has a current market capitalization of 450 billion, 40 billion in debt and 70 billion in cash. Assuming that the firm has a debt beta of 0.01 and an equity beta of 2.3, and that the cash is risk-free, estimate the beta of the firm's enterprise value.

Solution

We have $E = 450$, $D = 40$ and $C = 70$. The beta of the firm's enterprise value is

$$\beta_U = \frac{450}{450+40-70} \times 2.3 + \frac{40}{450+40-70} \times 0.01 = 2.465.$$

[END]

2.1.3 A Project's Cost of Capital

Weighted Average Cost of Capital (WACC)

If a project is of similar risk to a firm's core business activities, then it is reasonable to use the firm's equity cost of capital to value the project if the firm has no debt. But what if the firm uses debt financing? We can argue that the firm's asset and the project are of similar risk, and hence it seems reasonable to use r_U as the cost of capital. This is nearly correct, except one thing: in many countries, including the US, interest expense of a firm is tax-deductible. Let τ_c be the corporate tax rate. For \$1 interest payment made to the debtholders, \$$\tau_c$ is saved. As such, $r_D\,(1-\tau_c)$ is the after-tax debt cost of capital because the tax saving partly defrays the borrowing cost. While the asset cost of capital is $r_U = \frac{E}{E+D}r_E + \frac{D}{E+D}r_D$, the firm's effective after-tax cost of capital is

Weighted Cost of Capital

$$r_{wacc} = \frac{E}{E+D} r_E + \frac{D}{E+D} r_D (1-\tau_c) = r_U - \frac{D}{E+D} \tau_c r_D$$

In this regard, r_U is also called the pretax WACC.

Example (2.1.4)

You are given the following information for a firm:

(i) Debt: 5000 6% coupon bonds outstanding, 1000 par value, 25 years to maturity, selling for 105% of par; the bonds make annual payments.

(ii) Common stock: 75000 shares outstanding, selling for 60 per share; the beta is 1.10.

Also, the risk-free interest rate is 5% and the market risk premium is 7%.
Compute the weighted average cost of capital of the firm, assuming a corporate tax rate of 35%.

Solution

We first find the pretax cost of debt. Solving $1050 = 60\frac{1-(1+i)^{-25}}{i} + 1000(1+i)^{-25}$with the help of a financial calculator, we get $i = 5.6228\%$.

The cost of equity, according to CAPM, is $r_E = 5\% + 1.1 \times 7\% = 12.7\%$.

The market value of debt is $D = 5,000 \times 1050 = 5,250,000$.

The market value of equity is $E = 75,000 \times 60 = 4,500,000$. So, $D + E = 9,750,000$.

The WACC is $r_{wacc} = \frac{450}{975} \times 12.7\% + \frac{525}{975} \times 5.6228\% \times (1 - 0.35) = 7.8295\%$.

[END]

Industry Beta

If a new project is not similar to the firm's existing business, then one way to find the project's cost of capital is to find a firm whose main business is similar to the new project and estimate that firm's asset beta. Such a firm is called a comparison firm. If there are multiple comparison firms, we may even use the average of the comparison firms' asset costs of capital or asset betas. Many data vendors publish equity betas for main industries.

Example (2.1.5)

Marriott wants to compute the cost of capital for its new restaurant division. It has identified three comparison firms for its restaurant division. The table below provides the equity beta estimates, book values of debt, and market values of equity for the three comparison firms:

	β_E	Debt(billions)	Equity(billions)
Firm 1	0.75	0.004	0.096
Firm 2	1.00	2.300	7.700
Firm 3	1.08	0.210	0.790

The risk-free interest rate is 4%, the risk premium on the market portfolio is 8.4%, the CAPM holds, debts are risk-free, and corporate tax rate is zero. Estimate the cost of capital for Marriott's restaurant division, assuming that all three firms provide equally good comparisons for Marriott's restaurant division, and that the restaurant division is financed by equity only.

Solution

We have $\beta_D = 0$. So we can use $\beta_U = \frac{E}{E+D}\beta_E$ to compute the unlevered beta of each firm:

Firm 1: $\beta_U = \frac{0.096}{0.096+0.004} \times 0.75 = 0.72$,

Firm 2: $\beta_U = \frac{7.7}{7.7+2.3} \times 1.00 = 0.77$,

Firm 3: $\beta_U = \frac{0.79}{0.79+0.21} \times 1.08 = 0.8532$

We assume that the asset beta of Marriott's restaurant division can be approximated by

$$\frac{0.72 + 0.77 + 0.8532}{3} = 0.7811.$$

Since Marriott's restaurant division has no debt, its equity beta and asset beta are the same. By the CAPM, the cost of capital is 0.04+0.7811 × 0.084=10.56%.

[END]

Example (2.1.6)

Assume that Time-Warner is interested in acquiring the ABC television network from Disney. It has estimated the expected cash flows from acquiring ABC and wants to compute the cost of capital to value those cash flows. However, the major networks that are most comparable, NBC and CBS are owned by GE and Westinghouse, respectively, which have substantial cash flows from other courses. For these comparison firms, the table below presents hypothetical equity betas, debt to asset ratios, and the ratios of the market values of the network assets to all assets:

	Equity Beta	$D/(D+E)$	Network Assets / All Assets
GE	1.1	0.1	0.25
Westinghouse	1.3	0.4	0.50

Estimate the appropriate beta for the ABC acquisition. Assume that the debt of each of these comparison firms is risk-free, that the non-network assets of them are substantially similar.

Solution

First, we compute the asset beta of the two comparison firms:

$$\text{GE: } 0.9 \times 1.1 = 0.99, \text{ Westinghouse: } 0.6 \times 1.3 = 0.78$$

Viewing the asset side of the comparison firms as portfolios of network and non-network assets, we have

$$\begin{cases} 0.25\beta_N + 0.75\beta_{NA} = 0.99 \\ 0.50\beta_N + 0.50\beta_{NA} = 0.78 \end{cases}$$

and thus $\beta_N = 0.36$. This is the beta for the network asset.

[END]

Operating Leverage

Operating leverage is the relative proportion of fixed versus variable costs. A project with a relatively high proportion of fixed cost versus variable costs is said to have high operating leverage. Because the expenses of businesses with higher operating leverage do not proportionately increase as sales increase, the beta of a project with high operating leverage should be larger.

Execution Risk

Execution risk is the risk that a new project fails because of missteps in the firm's execution. Marketing mistakes and manufacturing delays are examples of execution risk. Some firms adjust for this risk by assigning a higher cost of capital to new projects. This is incorrect because execution risk is diversifiable and such risk should be taken into account by adjusting the expected cash flows of new projects.

Uses of the CAPM in Practice

When we calculate the NPV of a project, we need to estimate the project's future cash flows and also the cost of capital. You may wonder if the **CAPM** is a good model for projecting the cost of capital and the error involved (such as the estimation error of beta, the choice of the risk-free rate etc) is significant. It turns out that in general the errors in estimating the project's future cash flows have a far greater impact than the small discrepancies in the cost of capital. While not perfect, the CAPM is simple and robust. It does not allow a lot of leeway for managers to manipulate the cost of capital. Thus it is the predominant model used in practice.

Exercise 2.1

For Questions 1 to 3, consider the following information for bonds:

Rating	A or above	BBB	BB	B	CCC
Average default rate	0.2%	0.5%	2.3%	6.0%	12.2%
Average beta	0.05	0.1	0.18	0.25	0.30

1. A BB bond has a yield to maturity of 8% and an expected loss rate in the event of default of 65%. What is the expected return on the bond?

2. If the market risk premium is 5.5% and the risk-free interest rate is 3%, what is the expected return on a B bond?

3. There is a new bond issue that is BBB. You are given that

   (i) A B bond has an expected return of 5%.

   (ii) The market risk premium is 4.6%.

   (iii) The new BBB bond has an expected loss rate given default of 55

   Find the yield to maturity of the BBB bond.

   For Questions 4 to 6, refer also to the following apart from the table in Questions 1 to 3:

Firm	Debt rating	Equity beta	Market Capitalization	Total Enterprise Value
1	BBB	1.3	2000	8000
2	A	0.8	2400	4000
3	B	1.75	3000	4500

4. Estimate the asset beta of Firm 1. Repeat for Firm 2 and Firm 3.

5. Firm 1 and Firm 3 decide to merge into one firm. What is the asset beta for the conglomerate?

6. Firm 2 wins a law suit and receives 1000 cash. How would its asset beta change? Assume that the beta of cash is 0.02.

7. A company has a debt beta of 0.1 and an equity beta of 2.1. It has 20 million in debt outstanding, and the total enterprise value is 140 million.

   Calculate the pretax WACC of the company if the risk-free interest rate is 2% and the expected return on the market portfolio is 10%.

8. The standard deviation of the returns on the shares of DSD is 0.534% and the correlation coefficient between the returns on its shares and the return on the market is 0.789. The average return on the market is 14%, with a variance of $0.145\%^2$, and the risk-free rate is 6%.

   What is the cost of equity of the company?

9. A company's market value balance sheet is as follows:

Asset value	500	Debt	200
		Equity	300
Firm value	500	Firm value	500

The risk-free interest rate is 3.5%, the equity beta is 1.2, the debt beta is 0.2, and the return on market portfolio is 14.4%.

Calculate the company's cost of capital.

10. A company has an annual dividend yield of 7.5% and a constant dividend growth rate of 3% per year. It also has 5-year bonds outstanding that have an annual coupon rate of 8%, a redemption value of 1000 and are selling at a premium of 1025.

The company has a 35% corporate tax rate and a debt-to-asset ratio of 30%.

Calculate the firm's after-tax weighted average cost of capital.

11. A company is considering a project that will result in initial after-tax cash savings of $4 million at the end of the first year, and these savings will grow at a rate of 2% per year indefinitely. The firm has a target debt-value ratio of 50%, a cost of equity of 13%, and an after-tax cost of debt of 4.5%. The cost-saving proposal is somewhat riskier than the usual projects the firm undertakes; management uses the subjective approach and applies an adjustment factor of +2% to the cost of capital for such risky projects.

Find the cost of capital for the project.

12. You are given the following information about a company's expected returns and debt:

(i) Return on assets: 11%

(ii) Return on debt: 7%

(iii) The beta of the company's equity is 0.9.

Also, the market risk premium is 12.3%, and the risk-free interest rate is 4.5%.

If the company's debt amounts to 800, find the value of the company.

13. A firm has 30 million shares outstanding trading for 17 per share. It also has 90 million in outstanding debt. If the firm's equity cost of capital is 18.5%, debt cost of capital is 5.7%, and the corporate tax rate is 34%, find the firm's weighted average cost of capital.

14. Consider the following data from an industry:

Firm	Debt beta	Equity beta	Market Capitalization	Total Enterprise Value
1	0.05	2.3	1600	3500
2	0.30	1.8	2600	5200
3	0.06	2.0	3000	4500

Find the average asset beta for the industry based on the three firms.

15. A firm has 30 million shares outstanding trading for 20 per share. It also has 120 million in outstanding debt. If the firm's unlevered cost of capital is 22.5%, debt cost of capital is 7%, and the corporate tax rate is 40%, find the firm's weighted average cost of capital.

16. Which of the following statement(s) is/are true?

    I. Execution risk is the risk that a firm's new project fails because of an execution error.

    II. Execution risk should be handled by adjusting the expected cash flows of the project.

    III. A multi-divisional firm should use a single cost of capital for their projects.

    (A) I only

    (B) III only

    (C) I and II only

    (D) II and III only

    (E) I, II and III

17. You are given the following information about a firm:

    (i) Proportion of the firm's book value related to debt is 60%.

    (ii) Expected return on the firm's debt is 10%.

    (iii) Expected return on the firm's equity is 15%.

    (iv) The betas of the firm's debt, equity and asset are 0.3, 1.5 and 0.75, respectively.

    Find the firm's cost of capital.

Solutions to Exercise 2.1

1. The expected return is
YTM − P(default prob) × E(loss rate given default) $= 0.08 - 0.023 \times 0.65 = 6.505\%$.

2. The beta of the bond is 0.25.
By the CAPM, expected return $= 0.03 + 0.25 \times 0.055 = 4.375\%$.

3. For the B bond, applying the CAPM gives $5\% = r_f + 0.25 \times 4.6\%$. Hence $r_f = 3.85\%$.
For the BBB bond, we have, by the CAPM, $\mathrm{E}(R) = 0.0385 + 0.1 \times 4.6\% = 4.31\%$, and by the loss given default, $\mathrm{E}(R) = \mathrm{YTM} - 0.5\% \times 0.55$. Hence YTM $= 4.585\%$.

4. For Firm 1, the equity beta is 1.3, and the debt beta is 0.1. The asset beta is
$$\beta_U = \frac{E}{E+D}\beta_E + \frac{D}{E+D}\beta_D = \frac{1}{4} \times 1.3 + \frac{3}{4} \times 0.1 = 0.4$$
For Firm 2, $\beta_U = \frac{24}{40} \times 0.8 + \frac{16}{40} \times 0.05 = 0.5$
For Firm 3, $\beta_U = \frac{2}{3} \times 1.75 + \frac{1}{3} \times 0.25 = 1.25$

5. $\beta_U = \frac{8000}{12500} \times 0.4 + \frac{4500}{12500} \times 1.25 = 0.706$

6. $\beta_U = \frac{24}{30} \times 0.8 + \frac{16}{30} \times 0.05 + \frac{-10}{30} \times 0.02 = 0.66$

7. $\beta_U = \frac{120}{140} \times 2.1 + \frac{20}{140} \times 0.1 = 1.8143$
By the CAPM, the pretax WACC is $r_U = 0.02 + 1.8143 \times (0.1 - 0.02) = 16.51\%$.

8. The equity beta is
$$\frac{\mathrm{Cov}(R, R_M)}{\mathrm{Var}(R_M)} = \frac{\mathrm{Corr}(R, R_M)\mathrm{SD}(R)}{\mathrm{SD}(R_M)} = \frac{0.789 \times 0.534}{\sqrt{0.145}} = 1.10646$$
The cost of equity is $\mathrm{E}(R) = 0.06 + 1.10646(0.14 - 0.06) = 14.9\%$.

9. The debt to value ratio is $200/500 = 0.4$.
The beta of the company is $0.4 \times 0.2 + 0.6 \times 1.2 = 0.8$
Using the CAPM, the cost of capital is $0.035 + 0.8(0.144 - 0.035) = 12.22\%$.

10. By the dividend growth model, $r_E = \frac{\mathrm{Div}}{P} + g = 7.5\% + 3\% = 10.5\%$. Then we use a financial calculator to find the YTM of the bond. The equation is
$$1025 = 80(1 - v^5)/i + 1000v^5$$
giving $i = 7.384\%$. Hence, $r_{wacc} = 0.3 \times 0.07384 \times 0.65 + 0.7 \times 0.105 = 8.79\%$.

11. The debt-value ratio is 1/2. Thus, $\frac{D}{D+E} = \frac{1}{2}$ and $\frac{E}{D+E} = \frac{1}{2}$.
Hence $r_{wacc} = frac12 \times 0.13 = frac12 \times 0.045 = 0.0875$.
The appropriate discount rate for the project is $0.0875 + 0.02 = 0.1075$.

12. Applying the CAPM on the firm's asset, we have

$$0.11 = 0.045 + \beta_U \times 0.123,$$

and hence$\beta_U = 0.52846$. Then applying the CAPM on the firm's debt,

$$0.07 = 0.045 + \beta_D \times 0.123,$$

and hence$\beta_D = 0.20325$. Now by$\beta_U = \frac{800}{V} \times \beta_D + \frac{V-800}{V} \times \beta_E$,

$$0.52846V = 800 \times 0.20325 + (V - 800) \times 0.9$$
$$V = 1500.24$$

13. $E = 17 \times 30 = 510, D = 90, D + E = 600$
$r_{wacc} = \frac{510}{600} \times 0.185 + 0.057 \times (1 - 0.34) = 16.29\%$

14. We find the asset beta for each of the firms:
Firm 1: $\beta_U = \frac{16}{35} \times 2.3 + \frac{35-16}{35} \times 0.05 = 1.07857$
Firm 2: $\beta_U = \frac{26}{52} \times 1.8 + \frac{52-26}{52} \times 0.30 = 1.05$
Firm 3: $\beta_U = \frac{30}{45} \times 2.0 + \frac{45-30}{45} \times 0.06 = 1.35333$
The average asset beta is $(1.07857 + 1.05 + 1.35333)/3 = 1.16064$.

15. $E = 30 \times 20 = 600, D = 120, D + E = 720$
$r_{wacc} = 0.225 - \frac{120}{720} \times 0.07 \times 0.4 = 22.03\%$

16. (C)

III. The cost of capital for a particular line of business depends on the risk of that particular line.

17. We have $0.75 = 0.3w_D + 1.5(1 - w_D)$. Hence $w_D = 0.625$. So the cost of capital is $0.625 \times 0.1 + 0.375 \times 0.15 = 11.875\%$. Note that the proportion given in (i) is based on book value but not market value. Hence it is not correct to use 0.6 as the weight on debt.

Lesson 2

Risk Analysis

OBJECTIVES

1. To outline the steps in conducting a sensitivity analysis
2. To determine the break-even level of projects
3. To understand the concept of a Monte-Carlo simulation
4. To understand the use of a decision tree to summarize how the future would unfold and decisions should be made

In Exam FM, you have learnt **NPV** and **IRR** and used them to evaluate a project. In this lesson we are going to further explore techniques in project analysis when there are risk and uncertainty.

In this lesson we will base our discussion on the following case, which concerns Actex Learning's plan to offer a new course for actuaries. A detailed profit projection of the project is provided below:

	Time 0	End of Years 1 to 5
Revenue		40500
Variable costs		13500
Fixed costs		5000
Depreciation		1000
Pretax profit		21000
Tax (at 34%)		7140
Net profit		13860
Cash flow		14860
Initial investment	30000	

The cash flows at the ends of year 1 to 5 are calculated as follows:
Pretax profit $= 40500 - 13500 - 5000 - 1000 = 21000$
Corporate tax due to the project $= 21000 \times 0.34 = 7140$
Net profit $= 21000 - 7140 = 13860$

Cash flow $= 13860 + 1000 = 14860$ (Recall that depreciation is a non-cash expense. Read the appendix of this lesson if you are not familiar with this concept.)
The NPV is computed under a 20% **cost of capital** of the project:

$$\text{NPV} = -30000 + \frac{14860(1-1.2^{-5})}{0.2} = -30000 + 14860 \times 2.990612 = 14440.5.$$

It seems that the project is quite profitable.

2.2.1 Scenario and Sensitivity Analyses

The most important rule in capital budgeting is that a firm should choose a project with the highest NPV. To compute the NPV we need two things:

(a) future cash flows (to be precise, the *incremental* cash flows to the firm, computed by comparing the cash flows of the firm with and without the project);

(b) the **project's cost of capital**.

In many capital budgeting exercises, the cash flows in the future are subject to significant uncertainty. There can be many possible future scenarios and the NPV of a project is heavily dependent on which scenario would evolve in the future.

Typically there is a **base scenario** which serves as the average scenario based on management assumptions. The NPV of the project is computed based on this set of assumptions. This serves rather like the "mean".

NPV: Sensitivity Analysis

Sensitivity analysis breaks the NPV calculation into its component assumption and shows how the NPV varies as the underlying assumptions change. We recalculate the NPV under the worst- and best-case assumptions for each parameter. The resulting NPVs would reveal which assumption(s) is/are the most critical.

Example (2.2.1)

In the Actex Learning example, the revenue and variable costs are computed by the following formulas:

$$\text{Enrollment per year} = \text{Market size} \times \text{market share of Actex}$$

$$\text{Revenue} = \text{Enrollment per year} \times \text{price per unit}$$

$$\text{Variable cost} = \text{Variable cost per unit} \times \text{enrollment year}$$

The following table shows the best- and worst-case parameter assumptions for market size, market share, price per unit, variable cost per unit, fixed cost and initial investment.

Parameter	Initial assumption	Worst case	Best case
Market size	300	100	400
Market share	0.3	0.15	0.4
Price per unit	450	350	500
Variable cost per unit	150	180	140
Fixed cost	5000	6500	4000
Initial investment	30000	40000	25000

Conduct a sensitivity analysis for the project.

Solution

We vary the parameters one by one to the worst- and best-case scenario, keeping the other parameters fixed at their initially assumed values. For example, for market size, we first look at the worst case:

Number of enrollment $= 100 \times 0.3 = 30$, Revenue $= 30 \times 450 = 13500$

Variable cost $= 30 \times 150 = 4500$

Pretax profit $= 13500 - 4500 - 5000 - 1000 = 3000$

Net profit $= 3000 \times 0.66 = 1980$

Cash flow $= 1980 + 1000 = 2980$

NPV $= 2980 \times 2.990612 - 30000 = -21088$

Then we look at the best case:

Number of enrollment $= 400 \times 0.3 = 120$, Revenue $= 120 \times 450 = 54000$

Variable cost $= 120 \times 150 = 18000$

Pretax profit $= 54000 - 18000 - 5000 - 1000 = 30000$

Net profit $= 30000 \times 0.66 = 19800$

Cash flow $= 19800 + 1000 = 20800$

NPV $= 20800 \times 2.990612 - 30000 = 32204.73$

Repeating the same type of calculation for the other five parameters, we get the following table for NPVs:

Parameter	NPV for worst case	NPV for best case
Market size	−21088	32204.73
Market share	−12205.9	32204.73
Price per unit	−3323.74	23322.61
Variable cost per unit	9111.23	16216.92
Fixed cost	11479.79	16414.3
Initial investment	9440.50	19440.5

Obviously, the most crucial parameter is market size.

[END]

NPV: Scenario Analysis

In a sensitivity analysis, parameters are varied one at a time. However, in reality, certain parameters may move together. For example, it is possible that the market size is low, the market share and the price per unit would also be low. We can create a scenario when the market size, the market share and the price per unit are all low, and compute the corresponding NPV (other parameters unchanged) as follows:

Number of enrollment $= 100 \times 0.15 = 15$, Revenue $= 15 \times 350 = 5250$
Variable cost $= 150 \times 15 = 2250$
Pretax profit $= 5250 - 2250 - 5000 - 1000 = -3000$
Tax $= -3000 \times 0.34 = -1020$
(here we assume that Actex Learning has other projects that are profitable so that the loss on this project can offset the income elsewhere)
Net profit $= -3000 \times 0.66 = -1980$
Cash flow $= -1980 + 1000 = -980$
NPV $= -980 \times 2.990612 - 30000 = -32930.8$

A scenario analysis considers the effect of changing multiple parameters simultaneously.

2.2.2 Break-even Analysis

The **break-even** level of a parameter is the value of the parameter that makes the NPV zero, assuming all other parameters follow the initial assumptions. The IRR of the project is the break-even level of the cost of capital if it is treated as a parameter.

Example (2.2.2)

Calculate the break-even level for the following parameters for the Actex Learning example:

(a) Initial cost

(b) Variable cost per unit

Solution

(a) Since initial cost is the last parameter used in the calculation of NPV, we look at

$$\text{NPV} = -\text{Initial cost} + \frac{14860(1 - \frac{1}{1.2^5})}{0.2} = -\text{Initial cost} + 14860 \times 2.990612 = 0.$$

The initial cost is 44440.50.

(b) Number of enrollment $= 300 \times 0.3 = 90$, Revenue $= 90 \times 450 = 40500$
Variable cost $= 90\times$(Variable cost per unit)
Pretax profit $= 40500 - 90 \times$ (Variable cost per unit) $- 5000 - 1000$
$= 34500 - 90\times$(Variable cost per unit)
Net profit $= 0.66 \times [34500 - 90\times$(Variable cost per unit)]
$= 22770 - 59.4$(Variable cost per unit)
Cash flow $= 23770 - 59.4$(Variable cost per unit)
NPV$=-30000 + [23770 - 59.4$(Variable cost per unit)] $\times 2.990612 = 0$.

So the break-even variable cost per unit is 231.290.

[END]

2.2.3 Monte-Carlo Simulation

Sensitivity analyses and scenario analyses tell you "what if" one parameter or a set of parameters change. However, they do not tell you how likely the possible NPVs are. In a **Monte-Carlo simulation**, we use historical data (or judgment) to figure out possible values of the parameters and their associated probabilities. The final result is a frequency distribution of NPVs, from which a risk manager can estimate the probabilities of adverse and good outcomes.

The steps required to perform a Monte-Carlo simulation are as follows:

1. Model the NPV as a mathematical expression of the parameters
2. Specify the joint distribution of the parameters using historical data.
3. Use a computer to draw a scenario of the set of all parameters.
4. Calculate the future cash flows and the NPV that correspond to the scenario.
5. Repeat Steps 3 and 4 a very large number of times.
6. Plot the frequency distribution of the resulting NPVs. The mean of the distribution gives the "base case" NPV of the project, while the spread of the distribution indicates how risky the project is.

Scenario analyses are useful when the parameters are discrete because in such a situation we can list out all possible scenarios. For parameters that can vary in a continuous manner (such as future interest rates), a Monte-Carlo simulation would give us a fuller picture of the risk involved. However, it is difficult to use historical data to estimate the probability distribution of the parameters. Moreover, the true correlation structure between the parameters can be very complicated to model. Scenario analyses and Monte-Carlo simulations are always garbage in garbage out. A complex model may not necessarily provide the most accurate estimates.

Since Exam IFM is a multiple-choice exam and you are not required to know how Step 3 is performed, it is unlikely that you will be asked to conduct a Monte-Carlo simulation in Exam IFM.

2.2.4 Introduction to Decision Tree and Real Options

Many investment projects allow for decisions to be made in the future. For example, if the realized sales of the course after 2 years is good, Actex Learning may develop a second course to cover other topics. If the sales does not look promising after 2 years, Actex Learning may abandon the course after 4 years after all existing customers finish the course. Such opportunities are called **real options**. They can be summarized using a flowchart, which is known as a **decision tree**.

We can use the following principle to value real options:

> The market value of a project is the sum of the NPV of the project without options and the value of the real options implicit in the project.

The following examples illustrate how this general principle can be applied.

Example (2.2.3)

Consider a project which involves drilling an oil well. The drilling rig costs 300 today and in one year the well is either a success or a failure. If the outcome is a success, the PV of the payoff at time 1 will be 575. Otherwise, it will be 0. The outcomes are equally likely. The appropriate effective interest rate is 10% per annum.

(a) What is the NPV of the project?

(b) Instead of getting 0, the firm has an option to sell the rig for 250 at time 1 if the project fails.

(i) Compute the NPV of the project, taken into account the option to sell the rig.

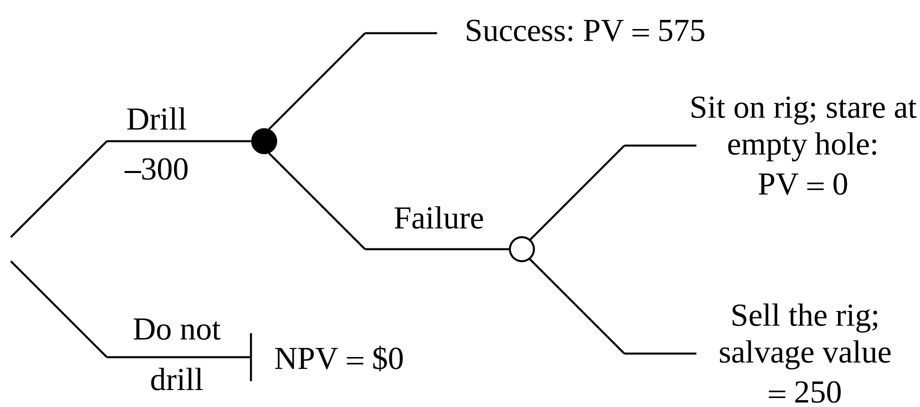
(ii) What is the value of the option to sell the rig?

Solution

(a) The expected value of the payoff at time 1 is$0.5 \times 575 + 0.5 \times 0 = 287.5$.

The NPV (at time 0) is therefore given by$-300 + 287.5/1.1 = -38.64$.

(b) Given the option to sell the rig, we have the following decision tree:



In the tree, circular nodes indicate the resolution of uncertainty, square nodes indicate decision point.

It is obvious that the firm should excise the option to sell the rig if the project fails at time 1. As a result, when we include the value of the option to abandon, the drilling project should yield an expected PV of $0.5 \times 575 + 0.5 \times 250 = 412.5$.

The revised NPV is given by $-300 + 412.5/1.1 = 75$.

We have$75 = -38.64+$value of option, which implies the value of option is 113.64.

[END]

Example (2.2.4)

We are examining a new project. We expect to sell 9,000 units per year at 35 net cash flow apiece for the next 10 years. In other words, the annual operating cash flow is projected to be $35 \times 9,000 = 315,000$. The appropriate effective interest rate is 16%, and the initial investment required is 1,350,000.

(a) What is the base-case NPV?

(b) After the first year, the project can be dismantled and sold for 950,000. If expected sales are revised based on the first year's performance, when would it make sense to abandon the investment? In other words, at what level of expected sales would it make sense to abandon the project?

(c) Explain how the 950,000 abandonment value can be viewed as the opportunity cost of keeping the project in one year.

Solution

(a) The base-case NPV is given by

$$9,000 \times 35\left(\frac{1}{1.16} + \frac{1}{1.16^2} + ... + \frac{1}{1.16^{10}}\right) - 1,350,000$$

$$= 315,000 \times \frac{1 - 1.16^{-10}}{0.16} - 1,350,000 = 172,466.66$$

(b) We abandon the project if the proceed from selling the equipment is greater than the present value of the future cash flows. Let Q be the breakeven sale quantity. We have:

$$950,000 = 35 \times Q \times \frac{1 - 1.16^{-9}}{0.16},$$

which gives $Q = 5,892.24$ Therefore, we abandon the project if the level of sales is less than 5,893 units.

(c) If we continue with the project in one year, we would forego the $950,000 that could have been used for something else.

[END]

The following three **real options** are commonly seen in capital budgeting:

1. **The option to expand**

   If we find a positive NPV project, we should consider the possibility of expanding the project to get a larger NPV. For example, the option to expand has value if demand turns out to be higher than expected. All other things being equal, we underestimate NPV if we ignore the option to expand.

2. **The option to abandon**

   The option to abandon has a value if demand turns out to be lower than expected. This type of real options is illustrated in Example (2.2.4).

3. **Timing options**

Timing options are valuable if the underlying variables are changing with a favorable trend.

In Module 6 Lesson 4 we will further discuss how real options can be valued.

Appendix

The **earnings before interest and tax** is defined as

$$\text{Earnings before interest and tax (EBIT)} = \text{Revenues} - \text{Cash Costs} - \text{Depreciation}.$$

The cost here can include both fixed costs and variable costs. Depreciation is an accounting concept. The firm may have acquired a machine at time 0 but the cost would be amortized over a period of time. Hence, EBIT does not represent the actual cash that the firm receives. It is important, though, because corporate tax is calculated based on EBIT. Suppose that the corporate tax rate is τ_c. After paying tax, we get the net profit

$$\text{Net profit (or unlevered net income)} = \text{EBIT}\times(1-\tau_c).$$

To get the amount of cash that can be distributed out, we add back the depreciation to the net profit and also subtract capital expenditure (which is the money that the firm spends to buy, maintain, or improve its fixed assets) and the increase in net working capital (net working capital is the difference between the firm's liquid assets and current liabilities):

$$\begin{aligned}\text{Free cash flow (FCF)} = \text{EBIT} \times (1-\tau_c) &+ \text{Depreciation} - \text{Capital expenditure}\\ &- \text{Increase in net working capital}\end{aligned}$$

When we calculate the NPV of a project, we project the increase in FCF in future years due to the project, and then discount them using the appropriate project's cost of capital.

Exercise 2.2

For Questions 1 to 4, consider Example (2.2.1).

1. Verify the values of the NPV in the sensitivity analysis for the market share cost component.

2. Verify the values of the NPV in the sensitivity analysis for the variable cost per unit share component.

3. Find the break-even level for market share.

4. Find the break-even level for variable cost per unit.

5. The investigation of the impact on NPV of changing multiple parameters is called

   (A) break-even analysis

   (B) sensitivity analysis

   (C) decision tree analysis

   (D) scenario analysis

   (E) Monte-Carlo simulation

6. The investigation of the distribution of the NPV by drawing scenarios from parameter probability distribution is called

   (A) break-even analysis

   (B) sensitivity analysis

   (C) decision tree analysis

   (D) scenario analysis

   (E) Monte-Carlo simulation

7. The difference between scenario analysis and sensitivity analysis is that

   (A) scenario analysis considers the value of multiple parameters for which the NPV is 0, while sensitivity analysis considers the value of each of the parameter for which the NPV is 0.

   (B) scenario analysis considers the effect of IRR on NPV, while sensitivity analysis does not.

   (C) scenario analysis considers the effect of multiple parameters on IRR, while sensitivity considers the effect of one single parameter on IRR.

   (D) scenario analysis considers the effect of multiple parameters on NPV, while sensitivity considers the effect of one single parameter on NPV.

   (E) scenario analysis is an alternative name for sensitivity analysis.

8. In a Monte-Carlo simulation,

   (A) parameters are assumed to be independent with each other.

   (B) the result of the simulation is the mean of the NPVs from all simulation runs.

   (C) the effect of parameters that vary in a discrete manner cannot be studied.

   (D) the effect of parameters that can vary continuously can be studied.

   (E) The correct answer is not given by (A), (B), (C) or (D).

9. Consider Example (2.2.4). Suppose that you think it is likely that expected sales from year 2 and so on will be revised upward to 11,000 units if the first year is a success and revised downward to 4,000 units if the first year is not a success. The annual sales in the first year is still $35 \times 9000 = \$315,000$. You think that success and failure are equally likely to occur after 1 year.

   (a) What is the NPV of the project if you cannot abandon the project?

   (b) What is the NPV of the project if you can abandon the project?

   (c) What is the value of the option to abandon?

Solutions to Exercise 2.2

1. We vary the market share parameter to the worst- and best-case scenario, keeping the other parameters fixed at the initial assumption. For the worst case:

   Number of enrollment $= 300 \times 0.15 = 45$, Revenue $= 45 \times 450 = 20250$
   Variable cost $= 45 \times 150 = 6750$
   Pretax profit $= 20250 - 6750 - 5000 - 1000 = 7500$
   Net profit $= 7500 \times 0.66 = 4950$
   Cash flow $= 4950 + 1000 = 5950$
   NPV $= 5950 \times 2.990612 - 30000 = -12205.8586$

   Then we look at the best case:

   Number of enrollment $= 300 \times 0.4 = 120$, Revenue $= 120 \times 450 = 54000$
   Variable cost $= 120 \times 150 = 18000$
   Pretax profit $= 54000 - 18000 - 5000 - 1000 = 30000$
   Net profit $= 30000 \times 0.66 = 19800$
   Cash flow $= 19800 + 1000 = 20800$
   NPV $= 20800 \times 2.990612 - 30000 = 32204.73$

2. We vary the variable cost parameter to the worst- and best-case scenario, keeping the other parameters fixed at the initial assumption. For the worst case:

   Number of enrollment $= 300 \times 0.3 = 90$, Revenue $= 90 \times 450 = 40500$
   Variable cost $= 90 \times 180 = 16200$
   Pretax profit $= 40500 - 16200 - 5000 - 1000 = 18300$
   Net profit $= 18300 \times 0.66 = 12078$
   Cash flow $= 12078 + 1000 = 13078$
   NPV $= 13078 \times 2.990612 - 30000 = 9111.22$

   Then we look at the best case:

   Number of enrollment $= 300 \times 0.3 = 90$, Revenue $= 90 \times 450 = 40500$
   Variable cost $= 90 \times 140 = 12600$
   Pretax profit $= 40500 - 12600 - 5000 - 1000 = 21900$
   Net profit $= 21900 \times 0.66 = 14454$
   Cash flow $= 14454 + 1000 = 15454$
   NPV $= 15454 \times 2.990612 - 30000 = 16216.92$

3. Let the break-even market share be b.
   Number of enrollment $= 300b$, Revenue $= 300b \times 450 = 135000b$
   Variable cost $= 300b \times 150 = 45000b$
   Pretax profit $= 135000b - 45000b - 5000 - 1000 = 90000b - 6000$
   Net profit $= (90000b - 6000) \times 0.66 = 59400b - 3960$
   Cash flow $= 59400b - 3960 + 1000 = 59400b - 2960$
   NPV $= (59400b - 2960) \times 2.990612 - 30000 = 0$
   On solving, we get $b = 21.8711\%$.

4. Let the break-even variable cost per unit be b.
Number of enrollment = 90, Revenue = 40500
Variable cost = $90b$
Pretax profit = $40500 - 90b - 5000 - 1000 = 34500 - 90b$
Net profit = $(34500 - 90b) \times 0.66 = 22770 - 59.4b$
Cash flow = $22770 - 59.4b + 1000 = 23770 - 59.4b$
NPV = $(23770 - 59.4b) \times 2.990612 - 30000 = 0$
On solving, we get$b = 231.2897$.

5. (D)

6. (E)

7. (D)

8. (D)

   (A) Parameters can be dependent. In this case the joint distribution of the parameters would be needed.

   (B) The result is a frequency plot of different NPVs. The means of all runs is only one statistic that can be computed from the frequency plot.

9. (a) If the project is a success, the PV of future cash flows at the end of year 1 will be

$$11,000 \times 35 \times \frac{1 - 1.16^{-9}}{0.16} = 1,773,519.39.$$

However, if the project is a failure, the PV of future cash flows at the end of year 1 will become

$$4,000 \times 35 \times \frac{1 - 1.16^{-9}}{0.16} = 644,916.14.$$

Now, the expected value of the project at the end of year 1 is given by

$$0.5 \times 1,773,519.39 + 0.5 \times 644,916.14 + 315,000 = 1,524,217.77.$$

And the NPV (at time 0) becomes $-\$1,350,000 + \$1,524,217.77/1.16 = -\$36,019.17$.

   (b) If we abandon the project at time 1, we would do so if sales after year 1 is 4000. Therefore, if the project is a failure, the cash flow at the end of year 1 is the salvage value of \$950,000. Since the scenarios are equally probable, the expected value of the project at the end of year 1 is given by

$$0.5 \times \$1,773,519.39 + 0.5 \times \$950,000 + \$315,000 = \$1,676,759.70.$$

As a result, the NPV (at time 0) is given by

$$-1,350,000 + 1,676,759.70/1.16 = 95,482.50.$$

   (c) The value of the option to abandon is $\$95,482.50 - (-\$36,019.17) = \$131,501.7$.

Lesson 3

Investment Risk Measures

OBJECTIVES

1. To recognize four risk measures
2. To calculate risk measures based on a probability distribution
3. To estimate risk measures from sample data
4. To understand the notion of coherence

In many financial institutions, **risk measures** are routinely calculated for the purpose of internal risk management. Many regulators also require companies such as banks and insurance companies to report risk measures. A risk measure may be used to determine the amount of asset to be kept in reserve. In this lesson, we are going to explore four different risk measures: variance, semi-variance, Value-at-Risk, and Tail-Value-at-Risk.

2.3.1 Variance and Semi-Variance

Let S be the price of a stock 1 year from now. Suppose that we have the following probability distribution for S:

s	12	14.5	15.5	17.75
$\Pr(S = s)$	0.2	0.25	0.35	0.2

The mean future stock price is

$$\mathrm{E}(S) = 12 \times 0.2 + 14.5 \times 0.25 + 15.5 \times 0.35 + 17.75 \times 0.2 = 15.$$

The **variance** of the future stock price is

$$\mathrm{E}[(S - 15)^2] = (-3)^2 \times 0.2 + (-0.5)^2 \times 0.25 + 0.5^2 \times 0.35 + 2.75^2 \times 0.2 = 3.4625.$$

The standard deviation is $\sqrt{3.4625} = 1.8608$.

Variance and standard deviation do not differentiate between upside risk and downside risk. As shown in the expression above, both upside risks

$$17.75 - 15 = 2.75 \text{ and } 15.5 - 15 = 0.5$$

and downside risks

$$14.5 - 15 = -0.5 \text{ and } 12 - 15 = -3$$

are involved in the average $\mathrm{E}[(S-15)^2]$. Upside risks for a stock investment do not represent adverse outcomes. To create a measure of downside risk, we can remove all of the terms that result in a positive deviation from the mean. This results in the **semi-variance**

$$\mathrm{E}\{[\min(0, S-15)]^2\} = (-3)^2 \times 0.2 + (-0.5)^2 \times 0.25 = 1.8625.$$

FORMULA

Semi-Variance

$$\sigma_{SV}^2 = \mathrm{E}\{[\min(0, S-\mu)]^2\} \text{ where } \mu = \mathrm{E}(S)$$

The positive square root of the above is known as the semi-standard deviation of S.

Example (2.3.1)

The return on a stock R follows a standard normal distribution. Find the semi-variance of R.

Solution

$R \sim \mathrm{N}(0,\ 1)$ and hence the density of R is $f_R(r) = \phi(r) = \frac{1}{\sqrt{2\pi}}\exp(-r^2/2)$. The mean of R is 0. So the semi-variance is $\mathrm{E}\{[\min(0,\ R)]^2\}$:

$$\begin{aligned}
\int_{-\infty}^{\infty} [\min(0, r)]^2 f_R(r)\mathrm{d}r &= \int_{-\infty}^{0} r^2 \frac{1}{\sqrt{2\pi}} e^{-r^2/2}\mathrm{d}r \\
&= \int_{-\infty}^{0} r \frac{1}{\sqrt{2\pi}} e^{-r^2/2}\mathrm{d}\frac{r^2}{2} = \int_{-\infty}^{0} r \frac{1}{\sqrt{2\pi}}\mathrm{d}e^{-r^2/2} \\
&= -\left[re^{-r^2/2}\right]_{-\infty}^{0} + \int_{-\infty}^{0} \frac{1}{\sqrt{2\pi}} e^{-r^2/2}\mathrm{d}r = 0 + \Pr(R<0) \\
&= 1/2
\end{aligned}$$

[END]

For stock prices or returns, downside risk refers to negative deviation and hence we look at negative values of

$$S - \mathrm{E}(S) \text{ or } R - \mathrm{E}(R).$$

If we consider a loss random variable X, then downside risk refers to positive deviation. The semi-variance would then be defined as

$$\sigma_{SV}^2 = \mathrm{E}\{[\max(0, X-\mu)]^2\} \text{ where } \mu = \mathrm{E}(X)$$

Make sure you understand the nature of the risk before you compute anything!

Example (2.3.2)

The operational loss X of a line of business can be modelled by an exponential distribution with mean θ. Find the semi-variance of X.

Solution

The mean of X is θ. So the semi-variance is $\mathrm{E}\{[\max(0,\ X-\theta)]^2\}$:

$$\begin{aligned}\int_{-\infty}^{0}[\max(0,x-\theta)]^2 f_X(x)\mathrm{d}x &= \int_{\theta}^{\infty}(x-\theta)^2\frac{1}{\theta}e^{-x/\theta}\mathrm{d}x \\ &= \int_{0}^{\infty} y^2\frac{1}{\theta}e^{-(y+\theta)/\theta}\mathrm{d}y \text{ (let } y=x-\theta) \\ &= e^{-1}\int_{0}^{\infty} y^2\frac{1}{\theta}e^{-y/\theta}\mathrm{d}y\end{aligned}$$

Note that the integral is the second moment of an exponential distribution with mean θ. Since $\mathrm{E}(X)=\theta$ and $\mathrm{Var}(X)=\theta^2$, the second moment is $2\theta^2$, and the semi-variance is $2e^{-1}\theta^2$.

[END]

In the discussion above we calculate semi-variance based on the true distribution of the underlying risk. If we only have a random sample of a risk (which may be obtained from simulation), then we can estimate the sample semi-variance by first estimating the sample mean, and then computing the corresponding negative deviations (if the sample is a collection of stock prices or returns) and finally taking an average.

Example (2.3.3)

You obtain the following sample of profit (in thousand dollars):

$$0.461, 0.201, 0.224, 0.052, 2.037, 0.489, 1.428, 2.877, 1.236, 0.818$$

Estimate the semi-variance of the profit.

Solution

The sample mean is $(0.461+0.201+\cdots+0.818)/10 = 0.9823$. The deviations are

$$-0.5213, -0.7813, -0.7583, -0.9303, 1.0547, -0.4933, 0.4457, 1.8947, 0.2537, -0.1643.$$

Since we are dealing with profit, we focus on the negative deviations. The sample semi-variance is the average of

$$0.5213^2, 0.7813^2, 0.7583^2, 0.9303^2, 0, 0.4933^2, 0, 0, 0, 0.1643^2.$$

Hence, we have

$$\frac{1}{10}(0.5213^2+0.7813^2+0.7583^2+0.9303^2+0.4933^2+0.1643^2) = 0.2593(\text{thousand dollar})^2.$$

[END]

2.3.2 Value-at-Risk and Tail-Value-at-Risk

Variance and semi-variance are functions of deviations. In this section we discuss two risk measures that are related to percentiles. We assume that the underlying risk is modelled by a continuous random variable X with a **cumulative distribution function** (cdf)

$$\mathrm{F}_X(x) = \Pr(X \le x).$$

The $100p$-th percentile is the unique solution of $\mathrm{F}_X(x) = p$. Defining F_X^{-1} as the inverse function of F_X, the solution of $\mathrm{F}_X(x) = p$ can be expressed as

$$100p\text{-th percentile of } X = \mathrm{F}_X^{-1}(p).$$

For example, if $X \sim \mathrm{N}(0, 1)$, the 50$^{\text{th}}$ percentile is $N^{-1}(0.5) = 0$, the 5$^{\text{th}}$ percentile is $N^{-1}(0.05) = -1.64485$, and the 10$^{\text{th}}$ percentile is $N^{-1}(0.10) = -1.28155$. (Note: In statistics we use $\Phi(z)$ for the cdf of N(0, 1). In finance and investment, we use $N(z)$ instead.)

Value-at-Risk (VaR)

When downside risk arises when realizations of X are small, we define the $100p\%$ Value-at-Risk as the solution of $\Pr(X \le x) = p$. So we have

$$100p\%\text{Value-at-Risk} = \mathrm{F}_X^{-1}(p).$$

Typically, p is chosen to be 0.05, 0.025, 0.01, or 0.005 (just like the significance level for a statistical test). If $p = 0.01$, then only 1 out of 100 times (on average) will the observed value of X be less than the VaR.

Example (2.3.4)

The returns on an investment follow a normal distribution with mean 7.5% and standard deviation 35%. Find the 10% Value-at-Risk of the investment return.

Solution

Let R be the random (one-year) investment return. Then $R \sim \mathrm{N}(0.075, 0.35^2)$. We have

$$\mathrm{F}_R(r) = \Pr(R \le r) = \Pr(Z \le \frac{r - 0.075}{0.35}) = N(\frac{r - 0.075}{0.35})$$

To find the $100p$-th percentile of R, we set $\mathrm{F}_R(r) = p$:

$$N(\frac{r - 0.075}{0.35}) = p,$$

and we have $r = 0.075 + 0.35N^{-1}(p)$. The 10% VaR is

$$0.075 + 0.35 \times (-1.28155) = -0.37354.$$

[END]

A very useful rule for computing the percentile of a function of random variable is the following:

> If $Y = g(X)$, where g is a continuously increasing function, then the 100pth percentile for Y is g(X_p), where $X_p = F^{-1}(p)$ is the 100pth-percentile for X.

Let us apply this result to redo Example (2.3.4). We write

$$R = 0.075 + 0.35Z,$$

where $Z \sim$ N(0, 1). Here $g(x) = 0.075 + 0.35x$ is strictly increasing. Since the 10^{th} percentile of Z is $N^{-1}(0.1) = -1.28155$, the 10^{th} percentile of R is $0.075 + 0.35 \times (-1.28155) = -0.37354$.

If g is a continuously decreasing function, then the 100p-th percentile of Y is $g(X_{1-p})$, where $X_{1-p} = F^{-1}(1-p)$.

When the adverse outcomes of X correspond to the realization of large values of X, p is chosen to be close to 1. Typical values are 0.95, 0.975, 0.99, and 0.995. The VaR can be conceived as the amount of capital required, with a high probability, to protect against insolvency.

Example (2.3.5)

The operational loss X of a line of business can be modelled by a single-parameter Pareto distribution with the following density function:

$$f(x) = \frac{\alpha\theta^\alpha}{x^{\alpha+1}}, x \leq \theta.$$

Find the 100p% Value-at-Risk of X.

Solution

The cumulative distribution function is $\mathrm{F}(x) = \int_\theta^x \frac{\alpha\theta^\alpha}{y^{\alpha+1}}\mathrm{d}y = 1 - \left(\frac{\theta}{x}\right)^\alpha, x \leq \theta$. We set $\mathrm{F}(x) = p$:

$$1 - \left(\frac{\theta}{x}\right)^\alpha = p.$$

The solution is $x = \frac{\theta}{(1-p)^{1/\alpha}}$.

[END]

If we only have a random sample of risks but not its true distribution, we can estimate the sample Value-at-Risk by first ranking the data in the sample and then "picking" the 100p-th percentile.

For example, if we have a sample with 11 data points, we can estimate the 50^{th} percentile, the 25^{th} percentile, and the 75^{th} percentile using the 6^{th}, 3^{rd} and the 9^{th} ranked data:

$$x(1), x(2), \underline{x(3)}, x(4), x(5), \underline{x(6)}, x(7), x(8), \underline{x(9)}, x(10), x(11),$$

where $x(1)$ denotes the smallest realization, and so on. However, it does not make much sense to estimate the 5% VaR from 11 points. What we know is that it should be less than the smallest observation $x(1)$! To estimate a 100p-th percentile for very small and very large p, a large sample is needed.

Example (2.3.6)

The following is the lowest 15 values of a sample of 150 weekly stock returns:

$$-0.9704, -0.5105, -0.4713, -0.4669, -0.4340,$$

$$-0.4226, -0.4185, -0.4105, -0.3986, -0.3634,$$

$$-0.3575, -0.3496, -0.3248, -0.3207, -0.3147.$$

Estimate the 5% Value-at-Risk of the weekly stock returns.

Solution

With a sample size of 150, and with $151 \times 0.05 = 7.55$, the 5% VaR should be in between the 7^{th} and the 8^{th} ranked data. We can do a linear interpolation as follows:

$$(-0.4185) \times 0.45 + (-0.4105) \times 0.55 = -0.4141,$$

or to be conservative, we can also take -0.4185 as the 5% VaR. The SoA Study Note says that the 7^{th} ranked data, the 8^{th} ranked data and the linear interpolated value are all acceptable answers.

[END]

Tail-Value-at-Risk (TVaR)

If a loss is greater than the 95% VaR, would it be close to the 99% VaR or would it be only slightly greater than the 95% VaR? One weakness of VaR is that it does not tell us how bad the result would be if we know that the outcome is more extreme than the VaR. One way to measure such "tail risk" is to look at the mean of the tail of the distribution. For $p > 0.5$ (for X being a loss) we can consider

$$\mathrm{E}(X|X > 100p\%\mathrm{VaR}) = \frac{1}{1-p}\int_{\mathrm{VaR}}^{\infty} x f_X(x)\mathrm{d}x,$$

and for $p < 0.5$ (for X being a profit or a return) we can consider

$$\mathrm{E}(X|X \leq 100p\%\mathrm{VaR}) = \frac{1}{p}\int_{-\infty}^{\mathrm{VaR}} x f_X(x)\mathrm{d}x.$$

Example (2.3.7)

A loss X follows a normal distribution with mean μ and standard deviation σ. Find the $100p\%$ Tail-Value-at-Risk of X for $p > 0.5$.

Solution

Following Example (2.3.4), it is easy to see that the $100p\%$ Value-at-Risk is $\mu + \sigma N^{-1}(p)$. Recalling that the density of the standard normal distribution is $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$, and that the density of $N(\mu, \sigma^2)$ is $\frac{1}{\sigma}\phi\left(\frac{x-\mu}{\sigma}\right)$, the $100p\%$ Tail-Value-at-Risk is

$$\begin{aligned}\frac{\int_{\mu+\sigma N^{-1}(p)}^{\infty} x\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})\mathrm{d}x}{1-p} &= \frac{1}{1-p}\int_{N^{-1}(p)}^{\infty}(\mu+\sigma z)\phi(z)\mathrm{d}z \\ &= \frac{1}{1-p}[\mu(1-p)+\sigma\int_{N^{-1}(p)}^{\infty} z\phi(z)\mathrm{d}z] \\ &= \frac{1}{1-p}[\mu(1-p)+\sigma\int_{N^{-1}(p)}^{\infty}\frac{e^{-z^2/2}}{\sqrt{2\pi}}\mathrm{d}\frac{z^2}{2}] \\ &= \mu+\frac{\sigma\phi[N^{-1}(p)]}{1-p}\end{aligned}$$

[END]

For most probability distributions, the formula for Tail-Value-at-Risk is hard to derive. In many cases it is easier to simulate a large sample of the underlying risk and then estimate the Tail-Value-at-Risk. The following is an example.

Example (2.3.8)

Consider the sample of 150 weekly stock returns in Example (2.3.6).

Estimate the 5% Tail-Value-at-Risk of the weekly stock returns.

Solution

The data below (or equal to) the 5% VaR are

$$-0.9704, -0.5105, -0.4713, -0.4669, -0.4340, -0.4226, -0.4185.$$

The TVaR can be estimated by the average of these 7 data points:

$$(-0.9704 - 0.5105 - \cdots - 0.4185)/7 = -0.527743.$$

[END]

TVaR is more conservative than VaR for the same value of p. This is because the VaR determines where the "tail" of a distribution of a risk begins, while the TVaR is the average of the whole tail.

TVaR is also called conditional tailed expectation (CTE) and expected shortfall (ES).

2.3.3 Coherent Risk Measure

It is useful to identify some desirable properties for risk measures. Let g be a risk measure for a random variable X. The following is the complete list of "desirable" properties that you need to know for Exam IFM (and also Exam STAM):

1. Translational Invariance

$$g(X+c) = g(X) + c$$

for a loss X. To understand the rationale behind this equality, treat $g(X)$ as the capital required to support the loss X. If the risk is increased by a constant amount c, then the capital required, which is $g(X+c)$, would also be increased by the same amount c.

If X is a return or the price of an asset, then the notion of translational invariance refers to

$$g(X+c) = g(X) - c.$$

This is why it is important to note the nature of X.

2. Positive Homogeneity

$$g(cX) = cg(X), \text{ for all } c > 0$$

Loosely speaking, if you double the loss, then the risk measure would also be doubled. This also implies that the risk measure is independent of the currency in which the risk is measured. Scaling up or down the risk does not provide risk diversification and hence the capital required would be scaled up or down in the same manner.

3. Subadditivity

$$g(X+Y) \leq g(X)_{+}g(Y)$$

This is the most important property. A risk measure should reflect the fact that aggregation should lower risk through diversification when the risks are less than perfectly correlated.

4. Monotonicity

$$\text{If } X \leq Y \text{ for all possible outcomes, then } g(X) \leq g(Y).$$

This gives an ordinal relation to risk.

A risk measure that satisfies all of the four properties above is called a **coherent risk measure**. It can be shown that

	Variance	Semi-Variance	VaR	TVaR
Translational Invariance	x	x	√	√
Positive Homogeneity	x	x	√	√
Subadditivity	x	x	x	√
Monotonicity	x	x	√	√

Hence only Tail-Value-at-Risk is a coherent risk measure among the four measures studied in the previous two sections.

Example (2.3.9)

Show the following by using appropriate examples:

(a) Variance does not satisfy monotonicity;

(b) Semi-variance does not satisfy translational invariance;

(c) Value-at-Risk does not satisfy subadditivity.

Solution

(a) Let X be a binomial random variable with parameters 1 and 0.5, and Y be always equal to 1. Then $X \leq Y$. However, $\text{Var}(X) = 0.25$ and $\text{Var}(Y) = 0$. So, $\text{Var}(X) > \text{Var}(Y)$.

(b) The random variable X defined in (a) has a semi-variance of $0.5 \times (0 - 0.5)^2 = 0.125$.

The possible outcomes of $X + 1$ are 1 and 2. They happen with equal probabilities.

The mean of $X + 1$ is $E(X) + 1 = 0.5 + 1 = 1.5$.
The semi-variance of $X + 1$ is $0.5 \times (1 - 1.5)^2 = 0.125$.
So the semi-variances of X and $X + 1$ are the same but not differ by 1.

(c) Consider an example related to operational risk. In a small town there are 2 stores A and B, and there is only one criminal living in the town. Every year, the probability that the criminal would rob neither A nor B is 0.92, the probability that he robs only A is 0.04, and the probability that he robs only B is 0.04. A robbery would cause a loss of 10000 to the store.

The 95% VaR of A is 0 because the loss of A is either 10000 with probability 0.04 or 0 with probability 0.96. Similarly, the 95% VaR of B is 0.

Now we add the losses of A and B. There is a probability of 0.92 that the total loss is 0, and a probability of 0.08 that the total loss is 10000, and hence the 95% VaR is 10000. So the VaR of the aggregate loss is greater than the sum of the individual VaRs.

[END]

Exercise 2.3

1. Let S be the stock price after 1 year. The distribution of S is as follows:

s	11	14	15.5	17
$\Pr(S=s)$	0.3	0.1	0.2	0.4

   Calculate the semi-variance of S.

2. Let X be the loss after 1 year. The distribution of X is as follows:

x	11	14	15.5	17
$\Pr(X=x)$	0.3	0.1	0.2	0.4

   Calculate the semi-variance of X.

3. The number of bonuses received by a sales agent is distributed as a geometric random variable with probability function

$$\mathrm{p}(x) = 0.75^x \times 0.25, x = 0, 1, 2, \ldots$$

   Calculate the semi-variance of the number of bonuses.

4. Let R be the annual simple return of a stock in the coming year. It is known that R can be modeled by a normal distribution with mean 0.14 and standard deviation 0.28.

   Calculate the semi-variance of R.

5. A risk measure that is similar to semi-variance is $\mathrm{E}\{|\min(0, S-\mu)|\}$ where μ=E(S). This is called the semi-mean absolute deviation of S.

   Calculate the semi-mean absolute deviation of S for the stock price in Question 1.

6. Show that the semi-mean absolute deviation defined in Question 5 satisfies positive homogeneity.

7. Show that variance does not satisfy subadditivity.

8. The following is the lowest 15 values of a sample of 100 weekly stock returns:

$$-0.7850, -0.5736, -0.5248, -0.4969, -0.4589,$$

$$-0.4533, -0.4230, -0.3859, -0.3844, -0.3784,$$

$$-0.3755, -0.3734, -0.3671, -0.3426, -0.3356.$$

   Estimate the 10% Value-at-Risk and 10% Tail-Value-at-Risk of the weekly stock returns.

9. Consider Example (2.3.5). Find the 100p% Tail-Value-at-Risk for $p > 0.5$and $\alpha > 1$.

10. Let X be a continuous loss random variable with density f. A risk measure g is defined as

$$g(X) = \mathrm{E}(X) + \sqrt{\int_{\mathrm{E}(X)}^{\infty} [x - E(X)]^2 f(x)\mathrm{d}x}.$$

(a) Is g translational invariant?
(b) Is g positive homogeneous?

11. Let R be the annual simple return of a stock in the coming year. It is known that R can be modeled by a normal distribution with mean 0.15 and standard deviation 0.35.

Calculate the 10% Value-at-Risk and 10% Tail-Value-at-Risk of R.

12. A coherent risk measure

(A) satisfies negative homogeneity.

(B) satisfies super-additivity.

(C) is concave.

(D) is rotational invariant.

(E) satisfies monotonicity.

13. (Related to Lesson 1 of Module 3) Let the annual return of a stock, R, be normally distributed with mean 0.3 and standard deviation 0.4, and let the time-1 stock price be $S = 50(1 + R)$.

Consider the following position of a loss

$$L = \begin{cases} -2.7 & 0 \leq S < 50 \\ S - 52.7 & S \geq 50 \end{cases}.$$

Find the Value-at-Risk of L at 97.5%.

14. (Related to Lesson 1 of Module 3) Let R be a N(0.2, 0.09) distributed risk, and $S = 50e^R$.

Consider the following position of an investment

$$P = \begin{cases} 2.5 & 0 \leq S < 50 \\ 52.5 - S & S \geq 50 \end{cases}.$$

Find the Value-at-Risk of P at 5%. If a capital cushion c is added to P, what is the minimum value of c such that the probability of P being less than 0 is less than or equal to 5%?

15. Let annual return R_1 be normally distributed with mean 0.3 and standard deviation 0.4, and annual return R_2 be normally distributed with mean 0.1 and standard deviation 0.3. R_1 and R_2 are bivariate normal, and the correlation between R_1 and R_2 is 0.4.

An investment has a time-1 value of $P = 50(1 + R_1) - 20(1 + R_2)$.

(a) Find $\Pr(P < 0)$.

(b) Find the semi-variance of P.

(c) Find the Value-at-Risk of P at 10%.

(d) Fine the Tail-Value-at-Risk of P at 10%.

Solutions to Exercise 2.3

1. $\text{E}(S) = 11 \times 0.3 + 14 \times 0.1 + 15.5 \times 0.2 + 17 \times 0.4 = 14.6$

   The semi-variance is $(11 - 14.6)^2 \times 0.3 + (14 - 14.6)^2 \times 0.1 = 3.924$.

2. $\text{E}(X) = 11 \times 0.3 + 14 \times 0.1 + 15.5 \times 0.2 + 17 \times 0.4 = 14.6$

   The semi-variance is $(15.5 - 14.6)^2 \times 0.2 + (17 - 14.6)^2 \times 0.4 = 2.466$.

3. The mean number of bonuses is $0.75/0.25 = 3$.

   $$p(0) = 0.25, p(1) = 0.75 \times 0.25 = 0.1875, p(2) = 0.75^2 \times 0.25 = 0.140625.$$

   The semi-variance is

   $$(0-3)^2 \times 0.25 + (1-3)^2 \times 0.1875 + (2-3)^2 \times 0.140625 = 3.140625.$$

4. We derive a general formula for the semi-variance of a normal distribution.

   $$R \sim \text{N}(\mu, \sigma^2) \text{ and hence } R = \mu + \sigma Z \text{ where } Z \sim \text{N}(0, 1).$$

   The mean of R is μ. So the semi-variance is

   $$\text{E}\{[\min(0, R - \mu)]^2\} = \text{E}\{[\min(0, \mu + \sigma Z - \mu)]^2\} = \sigma^2 \text{E}\{[\min(0, Z)]^2\} = \sigma^2 \times 0.5$$

   (where the last step follows from Example (2.3.1).)

   So the semi-variance of R is $0.282 \times 0.5 = 0.0392$.

5. The semi-mean absolute deviation is$|11 - 14.6| \times 0.3 + |14 - 14.6| \times 0.1 = 1.14$.

6. Let $c > 0$ be a constant and S be a random variable with mean μ. Then the semi-mean absolute deviation is

   $$\text{E}\{|\min(0, cS - \mu*)|\} \text{ where } \mu* = \text{E}(cS) = c\text{E}(S) = c\mu.$$

   Noting that $|\min(0,\ cS - c\mu)| = |c \min(0,\ S - \mu)| = c|\min(0,\ S - \mu)|$ because $c > 0$, we have

   $$\text{E}\{|\min(0, cS - \mu*)|\} = c\text{E}\{|\min(0, S - \mu)|\} = c \times \text{semi-mean absolute deviation of } S.$$

7. Let X be a random variable with variance 1, and $Y = 3X$. Then

   $\text{Var}(X)+\text{Var}(Y) = \text{Var}(X)+\text{Var}(3X) = \text{Var}(X) + 9\text{Var}(X) = 10\text{Var}(X) = 10$,

   $\text{Var}(X + Y) = \text{Var}(4X) = 16\text{Var}(X) = 16$,

   and hence $\text{Var}(X + Y) > \text{Var}(X)+\text{Var}(Y)$.

8. $101 \times 0.1 = 10.1$, and hence the 10% VaR is -0.3784 if we want to be conservative, or

   $$(-0.3784) \times 0.9 + (-0.3755) \times 0.1 = -0.37811,$$

   if we want to be slightly more accurate.

   The 10% Tail-VaR is the average of the 1st to the 10th ranked data:

   $$-0.1 \times (0.7850 + 0.5736 + 0.5248 + \cdots + 0.3784) = -0.48642.$$

9. $$\int_{\text{VaR}}^{\infty} x\frac{\alpha\theta^\alpha}{x^{\alpha+1}}\mathrm{d}x = \int_{\text{VaR}}^{\infty} \frac{\alpha\theta^\alpha}{x^{\alpha}}\mathrm{d}x = -\frac{\alpha\theta^\alpha}{\alpha-1}\left[x^{-\alpha+1}\right]_{\text{VaR}}^{\infty} = \frac{\alpha\theta^\alpha}{(\alpha-1)\ \text{VaR}^{\alpha-1}}$$

The $100p\%$ Tail-VaR is $\frac{\alpha\theta^\alpha}{(\alpha-1)\ \text{VaR}^{\alpha-1}} \times \frac{1}{1-p}$ where VaR $= \frac{\theta}{(1-p)^{1/\alpha}}$

$$= \frac{\alpha\theta^\alpha(1-p)^{\frac{\alpha-1}{\alpha}}}{(\alpha-1)\theta^{\alpha-1}} \times \frac{1}{1-p} = \frac{\alpha}{\alpha-1}\theta(1-p)^{-\frac{1}{\alpha}}$$

10. (a) We write $h(X) = \sqrt{\int_\mu^\infty (x-\mu)^2 f(x)\mathrm{d}x}$ where $\mu = \mathrm{E}(X)$. Let c be a constant. The mean of $X+c$ is $\mu + c$.

$$h(X+c) = \sqrt{\int_{\mu+c}^{\infty} (y-\mu-c)^2 f(y-c)\mathrm{d}y} = \sqrt{\int_{\mu}^{\infty} (y-\mu)^2 f(y)\mathrm{d}y} = h(X)$$

and hence $g(X+c) = \mathrm{E}(X+c) + h(X+c) = \mathrm{E}(X) + c + h(X) = c + g(X)$ So g is translational invariant.

(b) Let $c > 0$. The cdf of $Y = cX$ is $\Pr(Y \le y) = \Pr(X \le y/c) = \mathrm{F}(y/c)$. The density of Y is $f_Y(y) = \frac{1}{c}f(\frac{y}{c})$. Then $h(Y)$ is

$$\begin{aligned} h(cX) &= \sqrt{\int_{c\mu}^{\infty} (y-c\mu)^2 \frac{1}{c} f(\frac{y}{c}) dy} \\ &= \sqrt{\int_{\mu}^{\infty} (cu-c\mu)^2 \frac{1}{c} f(\frac{cu}{c}) c du} && (\text{ let } y = cu) \\ &= \sqrt{\int_{\mu}^{\infty} c^2 (u-\mu)^2 f(u) du} \\ &= c\sqrt{\int_{\mu}^{\infty} (x-\mu)^2 f(x) dx} && (let u = x) \\ &= ch(X) \end{aligned}$$

and hence g(cX) =E(cX)+h(cX) $= c\mathrm{E}(X) + \mathrm{ch}(X) = c[\mathrm{E}(X)+\mathrm{h}(X)] = cg(X)$.

So g is positive homogeneous.

Actually $g(X)$ is the sum of the mean of X and the semi-standard deviation of X.

11. In Example (2.3.7), we have obtained the following:

$$\int_{\mu+\sigma N^{-1}(p)}^{\infty} x\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})dx = \mu(1-p) + \sigma\phi[N^{-1}(p)]$$

As a result,

$$\begin{aligned} \int_{-\infty}^{\mu+\sigma N^{-1}(p)} x\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})dx &= \mathrm{E}(X) - \int_{\mu+\sigma N^{-1}(p)}^{\infty} x\frac{1}{\sigma}\phi(\frac{x-\mu}{\sigma})dx \\ &= \mu - \{\mu(1-p) + \sigma\phi[N^{-1}(p)]\} \\ &= \mu p - \sigma\phi[N^{-1}(p)] \end{aligned}$$

For $p < 0.5$, the Tail-VaR is $\frac{\mu p - \sigma\phi[N^{-1}(p)]}{p}$. For $p = 0.1$, the VaR is $0.15 + 0.35N^{-1}(0.1) = 0.15 - 0.35 \times 1.28155 = -0.29854$.

$\phi(-1.28155) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1.28155^2}{2}) = 0.175499$

The Tail-VaR is $\frac{\mu p - \sigma\phi[N^{-1}(p)]}{p} = 0.15 - \frac{0.35 \times 0.175499}{0.1} = -0.46425$.

12. (E)

    (A) It should be **positive** homogeneity.

    (B) It should be **sub**-additivity.

    (C) Coherence does not address if the transformation is convex or concave.

    (D) It should be **transitional** invariance.

13. L is increasing in S, and S is increasing in R.

    The 97.5^{th} percentile of R is $0.3 + 0.4 \times 1.9600 = 1.084$.

    The 97.5^{th} percentile of S is $50(1 + 1.084) = 104.2$.

    The 97.5^{th} percentile of L is $104.2 - 52.7 = 51.5$.

    Actually L is the loss of a market maker shorting selling a 50-strike call.

14. P is decreasing in S, and S is increasing in R. The 95^{th} percentile of R is $0.2 + 0.3 \times 1.64485 = 0.693455$.

    The 95^{th} percentile of S is $50e^{0.693455} = 100.0308$.

    The 5^{th} percentile of P is $52.5 - 100.0308 = -47.5308$.

    So, c should be 47.5308.

15. P is normally distributed, with $\text{E}(P) = 50(1 + 0.3) - 20(1 + 0.1) = 43$, and

$$\begin{aligned}\text{Var}(P) &= 50^2\ \text{Var}(R_1) + 20^2\ \text{Var}(R_2) - 2(50)(20)\text{Cov}(R_1, R_2) \\ &= 2500 \times 0.16 + 400 \times 0.09 - 2000 \times 0.4 \times 0.4 \times 0.3 = 340.\end{aligned}$$

    (a) $\Pr(P < 0) = \Pr(Z < -43/340^{0.5}) = N(-2.33200) = 0.00985 (\approx 0.01)$

    (b) Semi-variance $= 0.5\text{Var}(P)$ (refer to Question 4 for a proof) $= 0.5 \times 340 = 170$

    (c) The VaR is $43 - 340^{0.5} \times 1.28155 = 19.3694$.

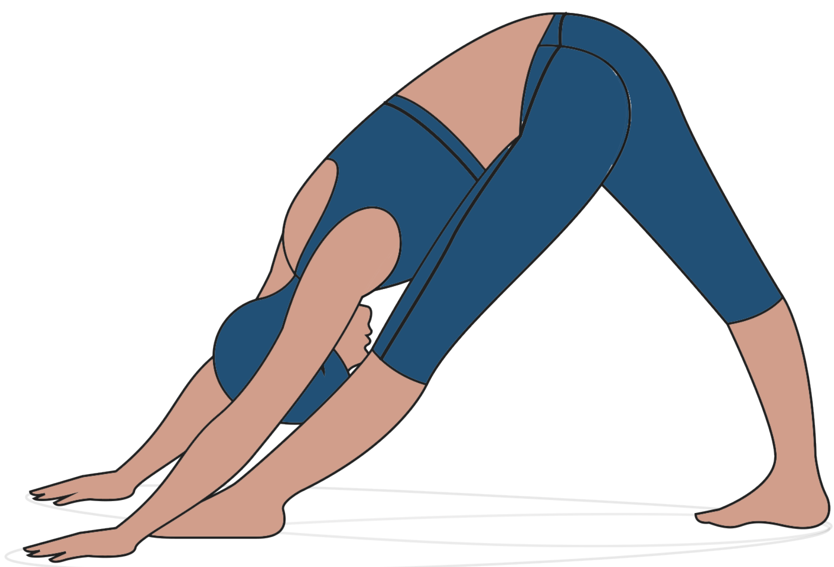
    (d) The Tail-VaR is $\frac{\mu p - \sigma\phi[N^{-1}(p)]}{p} = 43 - \frac{\sqrt{340}\phi(-1.28155)}{0.1} = 10.6396$.

STUDY BREAK

Do you have any fun ideas for future Study Breaks?

Share your inspiration with **#actexstudybreak**

Exercise is a proven way to boost your verbal memory, thinking, and learning. [1]



Pyramid with Arms Extended

Yoga - Sun Salutations

*Hold each pose for three long breaths, breathing in through your nose and out through your mouth.

Note that we are mathematicians, not yogi: attempt poses at your own risk!

[1]: https://www.health.harvard.edu/blog/regular-exercise-changes-brain-improve-memory-thinking-skills-201404097110

Exam IFM: General Information

The Investment and Financial Markets Exam is a three-hour exam that consists of 30 multiple-choice questions. Ideally, you should try to finish each question in about 6 minutes.

In the examination, you will be provided formulas that are related to normal and lognormal distributions. The formula sheet can be downloaded from:

https://www.soa.org/globalassets/assets/Files/Edu/2018/exam-ifm-cbt-table.pdf

and is also provided on the next page.

If you write a computer-based examination, you will be given a "normal distribution calculator." In this case, you should calculate normal probability values and z-values using that calculator.

If you write a paper-based examination, you will be provided a "normal table." In this case, you should calculate normal probability values and z-values using the normal table without doing any interpolation.

The options and solutions to the mock tests in this study guide are based on the normal distribution calculator.

Tables for Exam IFM

Unless otherwise stated in the examination question, assume:

- The market is frictionless. There are no taxes, transaction costs, bid/ask spreads or restrictions on short sales. All securities are perfectly divisible. Trading does not affect prices. Information is available to all investors simultaneously. Every investor acts rationally and there are no arbitrage opportunities.
- The risk-free interest rate is constant.
- The notation is the same as used in *Derivatives Markets*, by Robert L. McDonald.

In *Derivatives Markets*, $\Pr(Z \le x)$ is written as $N(x)$.

The standard normal density function is

$$f_Z(x) = N'(x) = \frac{e^{-x^2/2}}{\sqrt{2\pi}} = \frac{e^{-x^2/2}}{\sqrt{2 \times 3.14159}} = \frac{e^{-x^2/2}}{2.50663}, \quad -\infty < x < \infty.$$

Let Y be a lognormal random variable. Assume that $\ln(Y)$ has mean m and standard deviation v. Then, the density function of Y is

$$f_Y(x) = \frac{1}{xv\sqrt{2\pi}} \exp\left[-\frac{1}{2}\left(\frac{\ln(x) - m}{v}\right)^2\right], \quad x > 0.$$

The distribution function of Y is

$$F_Y(x) = N\left(\frac{\ln(x) - m}{v}\right), \quad x > 0.$$

Also,

$$\mathrm{E}[Y^k] = \exp\left(km + \frac{1}{2}k^2v^2\right),$$

which is the same as the moment-generating function of the random variable $\ln(Y)$ evaluated at the value k.

Formulas for Option Greeks:

Greeks	Call	Put
Delta (δ)	$e^{-\delta(T-t)}N(d_1)$	$e^{-\delta(T-t)}N(-d_1)$
Gamma (Γ)	$\dfrac{e^{-\delta(T-t)}N'(d_1)}{S\sigma\sqrt{T-t}}$	same as Γ for call
Theta (θ)	$\delta Se^{-\delta(T-t)}N(d_1) - rKe^{-r(T-t)}N(d_2) - \dfrac{Ke^{-\delta(T-t)}N'(d_2)\sigma}{2\sigma\sqrt{T-t}}$	call theta $+rKe^{-r(T-t)} - \delta Se^{-\delta(T-t)}$
Vega	$Se^{\delta(T-t)}N'(d_1)\sqrt{T-t}$	same as vega of call
Rho (ρ)	$(T-t)Ke^{-r(T-t)}N(d_2)$	$-(T-t)Ke^{-r(T-t)}N(-d_2)$
Psi (ψ)	$-(T-t)Se^{-\delta(T-t)}N(d_1)$	$(T-t)Se^{-\delta(T-t)}N(-d_1)$

Mock Test 1

1. Consider a European call option and a European put option on a nondividend-paying stock. You are given:

   (i) The current price of the stock is 50.

   (ii) The call option currently sells for 0.97 less than the put option.

   (iii) Both options will expire in 3 months.

   (iv) Both options have a strike price of 52.

   Calculate the continuously compounded risk-free interest rate.

   (A) 3.78%

   (B) 4.50%

   (C) 4.93%

   (D) 8.00%

   (E) 9.87%

2. Assume the Black-Scholes framework. For a stock whose price at time t is $S(t)$, which of the following statements is / are correct?

   I. $\mathrm{E}\left[\ln \dfrac{S(t+h)}{S(t)}\right]$ is proportional to h.

   II. $\mathrm{Var}\left[\ln \dfrac{S(t+h)}{S(t)}\right]$ is proportional to h.

   III. $\ln S(t)$ and $\ln \dfrac{S(t+h)}{S(t)}$ are independent random variables.

   (A) I only

   (B) III only

   (C) I and II only

   (D) II and III only

   (E) I, II and III

3. You are given:

(i) The current futures price of a stock index is F.

(ii) The stock index pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iii) The volatility of the futures price is 30%.

(iv) The continuously compounded risk-free interest rate is 6%.

(v) The price of a 9-month at-the-money European call on the futures is 9.88.

Calculate the price of a 9-month $(0.9F)$-strike European put on the futures.

(A) 5.36

(B) 5.61

(C) 12.04

(D) 12.47

(E) 14.92

4. You are given the following information about a portfolio consisting of stocks X, Y and Z:

Stock	Investment	Expected Return
X	10,000	8%
Y	$-15,000$	12%
Z	25,000	16%

Calculate the expected return of the portfolio.

(A) 14.2%

(B) 14.7%

(C) 15.0%

(D) 15.4%

(E) 15.8%

5. Consider a three-year project, where the cost of capital is 12%. There are only three cash flows for this project.

   – The first occurs at $t = 0$, and is -120.

   – The second occurs at $t = 1$, and is 66.

   – The third occurs at $t = 3$, and is c.

   Determine the value of c so that the project breaks even.

   (A) 76.6

   (B) 85.8

   (C) 93.4

   (D) 96.5

   (E) 100.2

6. You are given:

   (i) The price paths of the following stocks:

Stock	January	February	March	April
W	$23	$20	$22	$25
X	$34	$28	$35	$30
Y	$19	$28	$21	$22
Z	$42	$45	$36	$47

   (ii) None of the stocks pay dividends.

   (iii) You are an investor with the disposition effect.

   (iv) You bought each of these stocks in January.

   Suppose that it is currently the end of April. Identify the stock(s) you are inclined to hold on to.

   (A) X

   (B) Y

   (C) Y and Z

   (D) W, Y and Z

   (E) X, Y and Z

7. You are given:

(i) S_t represents the account value at time t.

(ii) T represents the future lifetime of the policyholder.

Consider a variable annuity with a GMDB that has a guarantee value of K. If the policyholder dies during the term of the variable annuity, the beneficiary will receive an amount of

(A) $S_T + \max(S_T - K, 0)$

(B) $\max(K - S_T, 0)$

(C) K

(D) $S_T + \max(K - S_T, 0)$

(E) S_T

8. You are given the following information about a portfolio that has two equally weighted stocks, X and Y.

(i) The economy over the next year could be good or bad.

(ii) The probability that the economy over the next year is good is 0.6.

(iii) The returns of the stocks can vary as shown in the table below:

Stock	Return when economy is good	Return when economy is bad
X	10%	−3%
Y	20%	−9%

Calculate the volatility of the portfolio return.

(A) 6.0%

(B) 8.6%

(C) 9.1%

(D) 10.3%

(E) 11.5%

9. For 4-month European options, you are given:

   (i) The underlying stock pays no dividends.

   (ii) The continuously compounded risk-free interest rate is 10%.

   (iii) The price of a 30-strike call is 1.5 higher than the price of a 32-strike call.

   Calculate the amount by which the price of an otherwise equivalent 32-strike put exceeds the price of an otherwise equivalent 30-strike put.

   (A) 0.34

   (B) 0.43

   (C) 0.52

   (D) 0.60

   (E) 0.68

10. Let r_D be the cost of debt of a firm, and τ_c be the firm's marginal corporate tax rate. Which of the following statements is/are true?

    I. The tax deductibility of interest lowers the effective cost of debt from r_D to $(1-\tau_c)r_D$.

    II. For every \$1 in new permanent debt that the firm issues, the value of the firm increases by $1-\tau_c$.

    III. The tax deductibility of interest lowers the weighted average cost of capital by $(1-\tau_c)r_D$.

    (A) I only

    (B) III only

    (C) I and II only

    (D) II and III only

    (E) I, II and III

11. Consider the banks A and B. Both banks have 50 loans outstanding. The principal on each of the loan is 1000, due today. Each loan has a default probability of 0.1. If defaults happen, the recovery value is only 600.

    For bank A, all loans concentrate in one industrial sector and the loans either all default or all not default. For bank B, the 50 loans are independent.

    Let σ_A be the standard deviation of the overall payoff to bank A, and σ_B be the standard deviation of the overall payoff to bank B.

    Find σ_A/σ_B.

    (A) 4

    (B) 5

    (C) 6

    (D) 7

    (E) 8

12. A market maker in stock index forward contracts observes a 1.5-year forward price of 117 on the index. The index spot price is 110 and the continuously compounded annual dividend yield on the index is 1%.

    The continuously compound risk-free interest rate is 5%.

    Describe actions that that market maker could take to exploit an arbitrage opportunity and calculate the resulting profit per index unit.

    (A) Buy synthetic forward, sell forward, profit = 0.05

    (B) Buy synthetic forward, sell forward, profit = 0.1

    (C) Buy synthetic forward, sell forward, profit = 0.2

    (D) Sell synthetic forward, buy forward, profit = 0.1

    (E) Sell synthetic forward, buy forward, profit = 0.05

13. Elaine decides to take a long position in 20 contracts of S&P 500 index futures. Each contract is for the delivery of 250 units of the index at a price of 1500 per unit, exactly one month from now. The initial margin is 5% of the notional value, and the maintenance margin is 90% of the initial margin. Elaine earns an annual continuously compounded interest rate of 4% on her margin balance. The position is marked-to-market on a daily basis.

    On the day of the first marking-to-market, the value of the index drops to 1498. On the day of the second marking-to-market, the value of the index is Y and Elaine is not required to add anything to the margin account.

    Calculate the minimum value of Y.

    (A) 1485.5

    (B) 1490.5

    (C) 1492.5

    (D) 1504.5

    (E) 1507.5

14. Which of the following observation(s) suggest(s) that investors have overreacted to new information?

    I. A few years after the IPO of a stock, the stock typically underperforms the market.

    II. There exists a positive serial correlation in stock prices.

    III. There exists a negative serial correlation in stock prices.

    (A) I only

    (B) II only

    (C) III only

    (D) I and II only

    (E) I and III only

15. For a stock, you are given

(i) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 5%.

(ii) The expected rate of appreciation of the stock is $a > 0$.

(iii) The volatility of the stock is 40%.

Find the probability that $S(1.5)$ is less than $S(0)e^{1.5a}$.

(A) $N(-0.398)$

(B) $N(-0.245)$

(C) $N(0.245)$

(D) $N(0.398)$

(E) There is not enough information.

16. Ben sells a 50-strike 1-year call on an index when the market price of the index is also 50. The call price is 4. Assume that the option is for an underlying 100 units of the index and the annual effective interest rate is 2%.

Calculate Ben's profit if the index increases to 52 at expiration.

(A) -408

(B) -208

(C) 0

(D) 208

(E) 408

17. For a stock, you are given:

(i) The current price of the stock is 50.

(ii) The volatility of the stock is 14.14%.

(iii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 1%.

(iv) The continuously compounded risk-free interest rate is 2%.

Calculate the current price of a 2-year at-the-money arithmetic average price call with a 2-period forward binomial tree.

(A) 3.78

(B) 4.03

(C) 4.89

(D) 5.32

(E) 6.47

18. A portfolio constructed by purchasing stocks of companies with small capitalizations and short-selling stocks of companies with large capitalizations is known as

(A) A PR1YR portfolio

(B) A PR2YR portfolio

(C) A HML portfolio

(D) A SMB portfolio

(E) A market portfolio

19. Determine which version of the efficient markets hypotheses is contradicted by a momentum strategy whereby investors can use past stock returns to form a portfolio with positive alpha.

(A) Weak form only

(B) Strong form only

(C) Weak form and semi-strong form only

(D) Strong form and semi-strong form only

(E) Weak form, semi-strong form, and strong form

20. You are given the historical month-end prices of the stock for the previous 6 months:

Month	**Stock Price**
1	100
2	102
3	105
4	104
5	102
6	103

Estimate the stock's volatility using the procedure described in *Derivatives Markets*.

(A) 0.02

(B) 0.04

(C) 0.07

(D) 0.09

(E) 0.11

21. You are given the following information about Stock X and the market:

(i) The risk-free interest rate is 4%.

(ii) The expected return and volatility for Stock X and the market are shown in the table below:

	Expected return	Volatility
Stock X	9.3%	45%
Market	12%	20%

(iii) The correlation between the returns of Stock X and the market is 0.3.

Assume the capital asset pricing model holds.

Which of the following statements is correct?

(A) The alpha of Stock X is negative, and the investor should invest in Stock X.

(B) The alpha of Stock X is negative, and the investor should not invest in Stock X.

(C) The alpha of Stock X is zero.

(D) The alpha of Stock X is positive, and the investor should invest in Stock X.

(E) The alpha of Stock X is positive, and the investor should not invest in Stock X.

22. For a nondividend-paying stock, you are given:

(i) The current price of the stock is 50.

(ii) In one year, the stock will either go up to 55 or down to 46.

(iii) The continuously compounded risk-free interest rate is 4%.

(iv) The current price of a 1-year 51-strike European call on the stock is 2.60.

Which of the following statement is true?

(A) You can make an arbitrage profit of 0.02 if you sell one unit of the call option, buy 0.444 shares of the stock and borrow 19.64.

(B) You can make an arbitrage profit of 0.02 if you buy one unit of the call option, sell 0.444 shares of the stock and lend 19.64.

(C) You can make an arbitrage profit of 0.43 if you sell one unit of the call option, buy 0.49 shares of the stock and borrow 10.29.

(D) You can make an arbitrage profit of 0.43 if you buy one unit of the call option, sell 0.49 shares of the stock and lend 10.29.

(E) There is no arbitrage opportunity.

23. You are given:

(i) The current price of a stock is 25.

(ii) The stock pays dividends continuously at a rate proportional to its price. The dividend yield is 3%.

(iii) The volatility of the stock is 0.3.

(iv) The continuously compounded risk-free interest rate is 0.04.

Find the elasticity for a 3-month 26-strike call option on the stock.

(A) 6.47

(B) 7.32

(C) 9.83

(D) 10.03

(E) 11.31

24. Let $S(t)$ and $Q(t)$ be the prices at time t of two stocks S and Q. You are given:

(i) $S(0) = Q(0) = 1$

(ii) The volatility of S is 40%, while the volatility of Q is 20%.

(iii) S pays no dividends.

(iv) Q pays dividends continuously at a rate proportional to its price. The dividend yield is 2%.

(v) The correlation between the returns on the two assets is -0.4.

An exchange option pays $\max[Q(1) - S(1), 0]$ at time 1.

Calculate the price of the exchange option.

(A) 0.15

(B) 0.16

(C) 0.17

(D) 0.18

(E) 0.19

25. Consider the following information on four stocks in a portfolio:

Stock	Weight	Beta
I	0.4	1.3
II	0.2	0.6
III	0.3	β_3
IV	?	1.8

The expected return on the portfolio is 12%. The risk-free interest rate is 4%, and the expected return on the market portfolio is 9%.

If the capital asset pricing model holds, find β_3.

(A) 1.6

(B) 1.8

(C) 2.0

(D) 2.3

(E) 2.6

26. You are given the following information about Landon Corporation:

(i) Landon is currently an all-equity firm.

(ii) Outside investors believe that the value of Landon will be either \$10 million (with a probability of 0.4) or \$5 million (with a probability of 0.6) in one year, depending on whether a new product of Landon is successful.

(iii) The CEO of Landon knows that the product will be successful and the value of Landon will be \$10 million in one year for sure. He wants outside investors to believe this.

Which of the following level of leverage can convince outside investors that the product will be successful?

(A) 2 million

(B) 3 million

(C) 4 million

(D) 5 million

(E) 6 million

27. Consider the following information concerning Landon Company:

   (i) It was originally funded by its founder Mary with an investment of $1 million.

   (ii) As of today, the initial investment represents 10,000,000 shares of Series A preferred stock.

   (iii) Today, Landon Company raises additional capital by selling equity in the form of convertible preferred stock, named Series B.

   (iv) In total, 5,000,000 shares of Series B preferred stock are sold at $0.75 per share.

   Calculate the sum of the post-money valuation and the pre-money valuation.

   (A) 16.75 million

   (B) 18.75 million

   (C) 20.75 million

   (D) 22.75 million

   (E) 24.75 million

28. Which of the following statements about coherent risk measure is/are correct?

   I. Subadditivity gives an ordinal relation to risk.

   II. Value-at-Risk is subadditive.

   III. Expected shortfall is subadditive.

   (A) I only

   (B) III only

   (C) I and II only

   (D) II and III only

   (E) None of the above

29. Which of the following statements about the capital market line is/are true?

   I. It is tangential to the efficient market frontier constructed from risky assets.

   II. It contains portfolios with the same beta.

   III. It passes through the risk-free asset.

   (A) II only

   (B) III only

   (C) I and II only

   (D) I and III only

   (E) II and III only

30. James has just sold 50 35-strike 6-month European call options on a nondividend-paying stock. You are given:

(i) The stock is currently trading at 32.

(ii) The stock's volatility is 35%.

(iii) The continuously compounded risk-free rate is 5%.

(iv) James delta hedges and does not rebalance his hedge portfolio.

If the stock price after 3 months stays at 32, and the call price becomes 1.26, calculate the 3-month profit.

(A) -4

(B) 21

(C) 35

(D) 44

(E) 47

*** End of Examination ***

Solutions to Mock Test 1

1.	D	11.	D	21.	B
2.	E	12.	C	22.	A
3.	A	13.	C	23.	C
4.	C	14.	E	24.	E
5.	B	15.	C	25.	E
6.	A	16.	D	26.	E
7.	D	17.	A	27.	B
8.	D	18.	D	28.	B
9.	B	19.	E	29.	D
10.	A	20.	C	30.	D

1. [Module 3 Lesson 3]

   This is a simple put-call parity problem. When there are no dividends, the put-call parity can be expressed as $c(K,T) - p(K,T) = S_0 - Ke^{-rT}$. So we have

   $$-0.97 = 50 - 52e^{-0.25r},$$

   which implies $r = 0.08$.

2. [Module 5 Lesson 1]

   Under the Black-Scholes framework,

   $$R(t, t+h) = \ln \frac{S(t+h)}{S(t)} \ N\left(\left(\alpha - \beta - \frac{\sigma^2}{2}\right)h, \sigma^2 h\right) \qquad \text{(a result of (5.1.7))}$$

   So, I and II are both correct.

   Similarly, $\ln S(t) = \ln S(0) + R(0,t)$ is independent of $R(t, t+h)$ because returns over two non-overalapping time periods are independent.

3. [Module 5 Lesson 2]

   Recall that the price of an at-the-money call on futures contract is

   $$c = p = Fe^{-rT}[2N(d_1) - 1],$$

   where $d_1 = -d2 = \dfrac{\sigma\sqrt{T}}{2}$. Plugging in the numbers given in the question, we have

   $d_1 = \dfrac{0.3\sqrt{0.75}}{2} = 0.1299038, N(d_1) = 0.55168$, and

   $$9.88 = Fe^{-0.06\times0.75}(2 \times 0.55168 - 1)$$

   giving $F = 99.9880$.