ACTEX Learning

Study Manual for Exam EA-1

Spring 2023 Edition

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An EA Exam

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Preface

Starting in May 2001, the Enrolled Actuaries Examinations, administered by the Joint Board for the Enrollment of Actuaries, were offered in a restructured form. The former series of three exams, denoted EA-1A, EA-1B, and EA-2, now appears as a new series of three exams denoted EA-1, EA-2F, and EA-2L. This ACTEX Study Manual has been prepared to assist the student's preparation for the EA-1 exam to be given in May 2023.

The subject matter of the EA-1 exam can be organized into two broad groups, as described in the Joint Board's Examination Program. The first group, denoted "mathematics of compound interest and financial analysis" in the Examination Program, does not require knowledge of probability or statistics. The second group, denoted "mathematics of life contingencies and demographic analysis" in the Examination Program, does require a familiarity with basic probably and statistics. Accordingly, this study guide is organized first into these two major sections, with each section split into two subsections, as follows:

Section	Topic
I-A	Compound Interest
I-B	Financial Analysis
II-A	Life Contingencies
II-B	Demographic Analysis

Then all of the EA-1 exam topics are organizing into units within each of these four sections, as shown in the Table of Contents. Each section begins with an Introductory Note describing the contents of each unit in that section.

The reader should understand that the anticipated content of the May 2023 EA-1 exam, upon which this ACTEX Study Manual is based, is based on the most recently published Joint Board Examination Program. In addition, we have included solutions to the 2001 through 2022 EA-1 exams. These 22 actual exams, with complete solutions, are presented in Section III of this manual. They should be viewed by the candidate as the best information available regarding what might be expected on the upcoming exam.

We hope this ACTEX Study Manual will be of great value to you as you prepare for the EA-1 exam. We would appreciate any feedback that you care to give us concerning the manual, including any errata that you find and any other suggestions for its improvement.

Michael J. Reilly, ASA, EA
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SECTION I-A - COMPOUND INTEREST

Introductory Note to Section I-A

The material summarized in the first two sections of this study manual addresses the mathematics of compound interest and financial analysis. It does not require a knowledge of probability and statistics.

The material in this section on traditional compound interest theory is organized into five units, as shown in the Table of Contents. For each unit of material, we (a) suggest one or more original textbook reference(s) from which the student can learn the material included in that unit, (b) present an overview commentary on the material, and (c) present a number of practice questions (with solutions) to test the student's under-standing of that unit's material. Because this material was included on past EA-1 exams, these unit practice questions have been taken from such past exams. Note that the sample questions are consecutively numbered through the entire manual.

The most recent Joint Board Examination Program lists several suggested readings that cover the interest theory material of this section. The principal readings are as follows:

1. Broverman, S.A., *Mathematics of Investment and Credit* (Sixth Edition). Winsted: ACTEX Publications, 2015.

2. Kellison, S.G., *The Theory of Interest* (Second Edition). Colorado Springs: Irwin/McGraw-Hill, 1991.

3. Parmenter, M.M., *Theory of Interest and Life Contingencies, with Pension Applications: A Problem-Solving Approach* (Third Edition). Winsted: ACTEX Publications, 1999.

These three references, along with a small study note mentioned in Unit I-A3, adequately cover all of the interest theory material that might reasonably be included on the EA-1 exam. The specific portions of these readings that relate to the material summarized in each unit of this section will be identified in the Overview Commentary presented in each unit.

Solutions manuals for the end-of-chapter exercises in each of these three texts are also available from ACTEX Publications.

Unit I-A1
Rates and Operations

Overview Commentary

A study of interest theory begins with an understanding of the types of rates used to define interest activity, and the types of operations that these rates are used to perform. These basic concepts are defined and illustrated in Chapter 1 of Broverman, Chapters 1 and 2 of Kellison, and Chapter 1 and Section 2.1 of Parmenter.

The student should understand the definitions of *effective rate of interest*, i, and *effective rate of discount*, d, and the several relationships between i and d; the definitions of *nominal rates of interest and discount*, denoted $i^{(m)}$ and $d^{(m)}$, and how they are converted to effective periodic rates; the *force of interest*, δ, and its relationship to effective and nominal rates; the distinction between the *compound interest* and *simple interest* growth functions; the operations of *accumulation* and *discount*, using either interest rates, discount rates, or forces, and using either the compound or simple growth patterns; and the very important concept of the *equation of value*.

In general, the force of interest is a function of time, and is properly denoted by δ_t. A special case arises when the force is a *constant* function of time, which occurs under the compound interest growth pattern. In this case, the subscript t is deleted, and the symbol δ is used. Although a constant force is by far the most common situation in practice, the student should be able to do accumulation and discount problems that involve a variable force of interest.

Questions involving *only* the material described in this section, with the possible exception of questions involving a variable force of interest, are necessarily fairly simplistic, and are therefore not common on the EA-1 exam. The types of questions that might realistically be expected are illustrated by the following sample questions.

SAMPLE QUESTIONS

1. Sole deposit to a fund: 1000 paid on 1/1/90
There have been no withdrawals from the fund.
Interest rate for 1990 through 1993: 8% per year, compounded quarterly.
Discount rate for 1994 through 1998: 6% per year, compounded every 4 months.
Force of interest for 1999 through 2001: 5% per year.

In what range is the value of the fund as of 1/1/2002?

A. Less than 2135
B. At least 2135 but less than 2145
C. At least 2145 but less than 2155
D. At least 2155 but less than 2165
E. 2165 or more

2. Deposit to fund: \$1300 paid on 1/1/95
Withdrawals from fund: None
Discount rate for 1995 through 2000: 7% per year, compounded quarterly
Interest rate for 2001 through 2005: 8% per year, compounded semiannually
Force of interest for 2006 through 2011: 6% per year

In what range is the value of the fund as of 1/1/2012?

(A) Less than \$4200
(B) \$4200 but less than \$4220
(C) \$4220 but less than \$4240
(D) \$4240 but less than \$4260
(E) \$4260 or more

3. Data for two funds:

	Fund A	Fund B
Interest/discount rate for first 10 years	$i^{(4)} = 6\%$	$d^{(12)} = 9\%$
Discount/interest rate for second 10 years	$d^{(4)} = 9\%$	$i^{(12)} = 12\%$
Initial amount in fund	$\$W$	$\$X$
Amount in fund at end of 20 years	$\$Y$	$\$Z$

There are no contributions to or withdrawals from either fund.

$\$W + \$X = \$10{,}000$ $\qquad$ $\$Y + \$Z = \$57{,}186$

In what range is $\$Y$?

(A) Less than \$27,500
(B) \$27,500 but less than \$28,400
(C) \$28,400 but less than \$29,300
(D) \$29,300 but less than \$30,200
(E) \$30,200 or more

4. Initial deposit to a fund: \$35,000

Withdrawal from the fund at the end of the fourth year: $70,000
Value of the fund at the end of the eighth year: $14,000
No other deposits or withdrawals were made during the eight-year period.

In what range is the annual rate of return for the fund during the eight-year period?

(A) Less than 14%
(B) 14% but less than 19%
(C) 19% but less than 24%
(D) 24% but less than 29%
(E) 29% or more

SOLUTIONS

S-1. The accumulated value on 1/1/2002 is given by
$AV = 1000\left(1+\frac{.08}{4}\right)^{16}\left(1-\frac{.06}{3}\right)^{-15}\cdot e^{3(.05)} = 1000(1.02)^{16}(.98)^{-15}(2.71828)^{.15} = 2159.51,$
ANSWER D.

S-2. The accumulated value of the fund after 17 years is given by
$AV = 1300\left(1-\frac{.07}{4}\right)^{-24}\left(1+\frac{.08}{2}\right)^{10}(e^{6(.06)}) = (1300)(.9825)^{-24}(1.04)^{10}(e^{.36}) = 4213.47,$
ANSWER B.

S-3. We are given the equations of value

$$W\left(1+\frac{.06}{4}\right)^{40}\left(1-\frac{.09}{4}\right)^{-40} = Y$$

and

$$X\left(1-\frac{.09}{12}\right)^{-120}\left(1+\frac{.12}{12}\right)^{120} = Z,$$

along with the relationships $W+X = 10{,}000$ and $Y+Z = 57{,}186$. The equations of value simplify to

$$4{,}50787W = Y$$

and

$$8.14523X = Z.$$

Adding we have

$$4.50787W + 8.14523(10{,}000-W) = Y+Z = 57{,}186$$

which solves for $W = 6671.40$. Finally

$$Y = 4.50787(6671.40) = 30{,}073.82, \text{ ANSWER D.}$$

S-4. The equation of value is

$$35{,}000(1+i)^8 - 70{,}000(1+i)^4 = 14{,}000.$$

Letting $x = (1+i)^4$ we have the quadratic equation

$$35x^2 - 70x - 14 = 0,$$

which solves for $x = \frac{70 + \sqrt{(70)^2 - 4(35)(-14)}}{2(35)} = 2.18322.$
Then $i = (2.18322)^{1/4} - 1 = .21555$, ANSWER C.

Unit I-A2
Annuities Certain

Overview Commentary

Annuities, defined as a series of periodic payments, are very important building blocks with many uses in the subject matter of this section of the EA-1 exam. The term *annuity certain* is used to distinguish interest-only annuities from *life annuities*, where payments are made contingent on the continued survival of one or more designated lives. They are described in Chapter 2 of Broverman, Chapter 3 of Parmenter, and Chapters 3 and 4 of Kellison.

The material reviewed in this unit includes the following: finding the present value and accumulated value of level-payment annuities, both *immediate* and *due*, using an effective rate of interest with compounding frequency the same as the payment frequency [$a_{\overline{n|}}$, $s_{\overline{n|}}$, $\ddot{a}_{\overline{n|}}$, $\ddot{s}_{\overline{n|}}$]; finding the present value of level-payment *perpetuities*, the limiting case of $a_{\overline{n|}}$ or $\ddot{a}_{\overline{n|}}$ as $n \to \infty$; finding the present value and accumulated value of non-level annuities (either *increasing* or *decreasing*), both immediate and due [$(Ia)_{\overline{n|}}$, $(Is)_{\overline{n|}}$, $(I\ddot{a})_{\overline{n|}}$, $(I\ddot{s})_{\overline{n|}}$, $(Da)_{\overline{n|}}$, $(Ds)_{\overline{n|}}$, $(D\ddot{a})_{\overline{n|}}$, $(D\ddot{s})_{\overline{n|}}$], and in the limiting case as $n \to \infty$, the present value of an increasing perpetuity.

Whenever the interest rate to be used in a present value or accumulated value calculation has a compounding frequency different from the frequency at which payments are made, a preliminary step is involved to simply find the effective rate, at the *payment* frequency, that is equivalent to the given rate. Both the Broverman and Parmenter texts take this approach. The Kellison text also takes this approach (see Section 4.2), but then also discusses an older approach, developed in the days before pocket calculators (see Sections 4.3 and 4.4). Kellison's older approach embodies some theoretical insights into annuity theory, but is not needed to obtain numerical results.

A special case of an annuity with payment made more frequently than interest is compounded is the abstract concept of the *continuous* annuity (see Section 2.2.3 of Broverman, Section 3.5 of Parmenter, or Section 4.5 of Kellison). Although the continuous annuity is of theoretical interest in certain cases, the Joint Board Examination Program suggests that it is not included on the EA-1 exam.

Although finding the present or accumulated value of a described sequence of payments is the most commonly asked question about annuities, there are also questions in which the present (or accumulated) value is known and the size of the payment, the number of payments, or the applicable interest rate must be determined.

The following sample questions illustrate the material reviewed in this unit.

SAMPLE QUESTIONS

5. Selected values: $s_{\overline{2n}|} = 43.7840$ $s_{\overline{3n}|} = 105.2974$ $(1+i)^n = 2.0803$

   In what range is $s_{\overline{n}|}$?

   (A) Less than 13.5
   (B) 13.5 but less than 14.0
   (C) 14.0 but less than 14.5
   (D) 14.5 but less than 15.0
   (E) 15.0 or more

6. Retirement benefit:
   $10, 000 per year payable annually commencing 12/31/88, increasing 3% per year compounded annually through 12/31/97 and then level for the remaining period.
   Form of payment: 15 years certain
   Assumed interest rate: 6%

   In what range is the present value of the retirement benefit as of 1/1/88?

   (A) Less than $114, 000
   (B) $114, 000 but less than $116, 000
   (C) $116, 000 but less than $118, 000
   (D) $118, 000 but less than $120, 000
   (E) $120, 000 or more

7. Retirement benefit: $1, 000 per year, payable annually commencing 1/1/88
   Form of payment: Perpetuity
   Assumed interest rate:
   Years 1-10: 8%
   After year 10: 6%

   In what range is the present value of the retirement benefit as of 1/1/88?

   (A) Less than $15, 000
   (B) $15, 000 but less than $15, 500
   (C) $15, 500 but less than $16, 000
   (D) $16, 000 but less than $16, 500
   (E) $16, 500 or more

8. Initial deposit to guaranteed investment contract: $100, 000
   Purchase date: 1/1/88
   Maturity date: 1/1/98
   Interest credited on initial deposit: 8% per year, reinvested at the end of each year
   Interest credited on additions: 6% per year, reinvested at the end of each year

   In what range is the accumulated value of the contract as of 1/1/98?

   (A) Less than $180, 000
   (B) $180, 000 but less than $195, 000
   (C) $195, 000 but less than $210, 000
   (D) $210, 000 but less than $225, 000
   (E) $225, 000 or more

9. A pension fund consists of Accounts A and B:

Market value of Account A as of 1/1/89:	\$200,000
Interest on Account A:	6% per year, credited on 12/31
Market value of Account B as of 1/1/89:	\$100,000
Interest on Account B:	9% per year, compounded semiannually, credited on 6/30 and 12/31

Each interest payment from Account A is immediately invested in Account B. There have been no contributions to or disbursements from the pension fund since 1/1/89.
In what range is the market value of the pension fund as of 1/1/94?

(A) Less than \$422,000
(B) \$422,000 but less than \$424,000
(C) \$424,000 but less than \$426,000
(D) \$426,000 but less than \$428,000
(E) \$428,000 or more

10. Type of plan: Money purchase
Plan effective date: 1/1/89
Employer contribution: 10% of actual compensation for the year

Allocation date:
December 31 of each year.
No allocation is made in the calendar year in which a participant retires, but the participant earns interest on his account balance until age 65.

Actuarial assumptions:
Interest rate: 7%
Compensation increases: 5% per year

Data for sole participant:
Date of birth: 7/1/49
Date of hire: 7/1/78
1988 compensation: \$40,000

In what range is the projected account balance at age 65?

(A) Less than \$360,000
(B) \$360,000 but less than \$400,000
(C) \$400,000 but less than \$440,000
(D) \$440,000 but less than \$480,000
(E) \$480,000 or more

11. Annuity A: 1000 per year payable at the end of each year for n years, with no deferral period.

Annuity B: 2000 per year, payable at the end of each year for $2n$ years, with a deferral period of m years.

Annuity C: Level annual amount, payable at the end of each year for $(m+n)$ years, with no deferral period.

Annuity C is equivalent in value to Annuities A and B combined.

Selected annuity values: $a_{\overline{m}|} = 8.273$ $\quad a_{\overline{n}|} = 11.101$ $\quad a_{\overline{m+n}|} = 12.780$

In what range is the annual payment for Annuity C?

A. Less than 1675
B. At least 1675 but less than 1700
C. At least 1700 but less than 1725
D. At least 1725 but less than 1750
E. 1750 or more

12. Selected annuity values: $\ddot{a}_{\overline{t}|} = 7.452$ $\quad \ddot{a}_{\overline{t+1}|} = 7.950$
In what range is $\ddot{s}_{\overline{25}|}$?

A. Less than 59
B. At least 59 but less than 64
C. At least 64 but less than 69
D. At least 69 but less than 74
E. 74 or more

13. Date of offering of a perpetuity: 1/1/91
Dividend dates: 3/31, 6/30, 9/30, and 12/31 each year
Amount of dividend each quarter in 1991: 1
The quarterly dividend each year is 8% greater than the prior year's quarterly dividend
Purchaser's yield rate: 10% per year, compounded annually

In what range is the purchase price of the perpetuity as of 1/1/91?

A. Less than 207
B. 207 but less than 212
C. 212 but less than 217
D. 217 but less than 222
E. 222 or more

14. Effective date of a decreasing annuity: 1/1/91
Date of first payment: 4/1/91
Frequency of payments: Quarterly
Number of payments: 40
Quarterly decrease in payment: 300
Interest rate: 8% per year, compounded quarterly

Present value of remaining payments as of 1/1/96 (after 1/1/96 payment is made): 50,000

In what range is the first quarterly payment?

A. Less than 11,500
B. 11,500 but less than 11,700
C. 11,700 but less than 11,900
D. 11,900 but less than 12,100
E. 12,100 or more

15. Effective date of annuity: 1/1/92
Date of first payment: 3/31/92
Frequency of payments: Quarterly
Number of payments: 40
Schedule of payments:

Date	Amount
Each 3/31	1
Each 6/30	2
Each 9/30	3
Each 12/31	4

Interest rate: 7% per year, compounded annually

In what range is the present value of the annuity as of 1/1/92?

A. Less than 63
B. 63 but less than 66
C. 66 but less than 69
D. 69 but less than 72
E. 72 or more

16. Effective date of perpetuity: 1/1/92
Interest rate: 8% per year, compounded annually

Payment schedule:

Date	Amount
1/1/92	10
1/1/93	30
1/1/94	50
1/1/95	70
1/1/96	90
1/1/97 and each 1/1 thereafter	110

In what range is the present value of the perpetuity as of 1/1/92?

A. Less than 1110
B. 1110 but less than 1210
C. 1210 but less than 1310
D. 1310 but less than 1410
E. 1410 or more

17. Effective date of an annuity certain: 1/1/93

    Date of first payment: 1/1/93
    Frequency of payments: Annual
    Amount of each payment: \$50,000
    Number of payments: 20

    Effective date of a perpetuity: 1/1/93

    Date of first payment: 1/1/93
    Frequency of payments: Monthly
    Amount of each payment: \$$X$

    Interest rate: 8% per year, compounded semiannually

    The perpetuity is actuarially equivalent to the annuity certain.

    In what range is \$$X$?

    (A) Less than \$3,380
    (B) \$3,380 but less than \$3,430
    (C) \$3,430 but less than \$3,480
    (D) \$3,480 but less than \$3,530
    (E) \$3,530 or more

18. Terms of a perpetuity:
    Issue date : 1/1/93
    Date of first payment: 12/31/93
    Frequency of payments: Annual
    Amount of first payment: \$500
    Increase in subsequent payments: 5% per year, compounded annually
    Interest rate: 8% per year, compounded annually

    In what range is the present value of the perpetuity as of 1/1/93?

    (A) Less than \$14,000
    (B) \$14,000 but less than \$15,000
    (C) \$15,000 but less than \$16,000
    (D) \$16,000 but less than \$17,000
    (E) \$17,000 or more

19. As of 1/1/94, the present value of an increasing perpetuity with annual payments of \$1, \$3, \$5, \$7, … payable each 1/1 commencing 1/1/94 is 25 times the present value of a level perpetuity with annual payments of \$1 payable each 1/1 commencing 1/1/94.

    In what range is the effective annual rate of interest?

    (A) Less than 7%
    (B) 7% but less than 8%
    (C) 8% but less than 9%
    (D) 9% but less than 10%
    (E) 10% or more

20. $1,000,000 is deposited on 1/1/94 to provide the following annuity:

    A payment each 12/31 from 12/31/94 through 12/31/2023 which will increase by 4% annually,

    plus

    a payment on 12/31/2023 equal to $1,000,000 accumulated from 1/1/94 at 2% per year, compounded annually.

    Interest rate: 10% per year, compounded semiannually

    In what range is the total payment due on 12/31/2023?

    (A) Less than $2,000,000
    (B) $2,000,000 but less than $2,010,000
    (C) $2,010,000 but less than $2,020,000
    (D) $2,020,000 but less than $2,030,000
    (E) $2,030,000 or more

21. Present value of an increasing monthly perpetuity as of 1/1/94: $320,000
    Payments: $10 commencing 1/1/94, increasing by $10 each month thereafter
    Interest rate: *X*% per year, compounded annually (*X* is greater than zero)

    In what range is *X*%?

    (A) Less than 6.70%
    (B) 6.70% but less than 6.80%
    (C) 6.80% but less than 6.90%
    (D) 6.90% but less than 7.00%
    (E) 7.00% or more

22. Terms of an annuity:
    Date of first payment: 12/31/95
    Frequency of payments: Annual, at the end of each year
    Amount of payments:
    $1500 per year, payable for the first n years
    $2500 per year, payable for the next m years
    $3500 per year, payable for the next $2n$ years

    Selected annuity values:

    $$a_{\overline{n|}} = 8.559 \qquad a_{\overline{m|}} = 9.818 \qquad v^m = .215$$

    In what range is the present value of the annuity as of 1/1/95?

    (A) Less than $18,000
    (B) $18,000 but less than $19,500
    (C) $19,500 but less than $21,000
    (D) $21,000 but less than $22,500
    (E) $22,500 or more

23. Effective date of a perpetuity: 1/1/95
Date of first payment: 12/31/95
Frequency of payments: Annual
Annual payment: $100 in 1995. Each of the succeeding 9 payments is twice the previous payment. Payments remain at $51,200 thereafter.
Interest rate: 7% per year, compounded annually
In what range is the present value of the perpetuity as of 1/1/95?

(A) Less than $425,000
(B) $425,000 but less than $435,000
(C) $435,000 but less than $445,000
(D) $445,000 but less than $455,000
(E) $455,000 or more

24. List price of a car: $20,000
Purchase option I: Immediate payment of list price less rebate of $*X*
Purchase option II: $500 down payment by purchaser
Level monthly payment at the end of each of the next 60 months
Interest rate: 6% per year, compounded monthly

X is determined so that, at an interest rate of 9% per year compounded monthly, the two options are financially equivalent.

In what range is the value of *X*?

(A) Less than $1350
(B) $1350 but less than $1400
(C) $1400 but less than $1450
(D) $1450 but less than $1500
(E) $1500 or more

25. Repayment schedule for a loan:

$*N* payable on January 1 of each year from 1995 through 2009, and an additional $*N* payable on January 1 of each of the following years: 1996, 1999, 2002, 2005, and 2008.

Present value of all future repayments:

As of 1/1/94: $*P*
As of 1/1/97: $*P* – $11,061 (prior to 1/1/97 repayment)

Interest rate: 7% per year, compounded annually

In what range is $*N*?

(A) Less than 4,500
(B) 4,500 but less than 9,000
(C) 9,000 but less than 13,500
(D) 13,500 but less than $18,000
(E) 18,000 or more

26. Date of first payment of a perpetuity: 1/1/97

Amount of each payment: $\frac{\$(n+1)(n+2)}{2}$, where $n = 0$ at 1/1/97 and increases by 1 each 1/1 thereafter

Interest rate: 25% per year, compounded annually

In what range is the present value of the perpetuity as of 1/1/97?

(A) Less than \$70
(B) \$70 but less than \$90
(C) \$90 but less than \$110
(D) \$110 but less than \$130
(E) \$130 or more

27. Frequency of deposits to a savings account: Monthly
Date of first deposit: 1/31/81
Amount of each deposit: \$25 each month in first year, increasing each January 31 thereafter by 12% over the monthly amount for the prior year
Interest rate: 12% per year, compounded monthly

In what range is the value of the savings account as of 1/1/99?

(A) Less than \$40,300
(B) \$40,300 but less than \$40,700
(C) \$40,700 but less than \$41,100
(D) \$41,100 but less than \$41,500
(E) \$41,500 or more

28. At age 30, Smith established a savings account with an initial deposit of \$5,500. He determined that by making 30 additional annual deposits of \$5,500 at each subsequent age he would accumulate \$1,000,000 in the savings account at age 61.

At age 45, the annual rate of return from age 30 to age 45 is determined to be 9% per year, compounded annually. If he continues to earn this rate of return to age 61, he will not accumulate \$1,000,000 at age 61.

Beginning with the deposit made at age 45, Smith changes the 16 remaining annual deposits to $\$X$ in order to accumulate \$1,000,000 in the savings account at age 61. He assumes the annual rate of return continues to be 9% per year.

In what range is $\$X$?

(A) Less than \$7,000
(B) \$7,000 but less than \$8,000
(C) \$8,000 but less than \$9,000
(D) \$9,000 but less than \$10,000
(E) \$10,000 or more

29. Terms of an annuity:

    Date of first payment: 1/1/99
    Frequency of payments: Monthly
    Amount of each payment:
        First 5 years: $500 per month
        Next 5 years: $650 per month
        Final payment: $10,000 on 1/1/2009

    Interest rate: 7% per year, compounded annually

    In what range is the present value of the annuity as of 1/1/98?

    (A) Less than $50,500
    (B) $50,500 but less than $51,500
    (C) $51,500 but less than $52,500
    (D) $52,500 but less than $53,500
    (E) $53,500 or more

30. Selected annuity values:

    $\ddot{a}_{\overline{n+2|}} = 14.030$ $\qquad$ $\ddot{s}_{\overline{n|}} = 52.344$

    In what range is the effective annual interest rate?

    (A) Less than 5.00%
    (B) 5.00% but less than 5.25%
    (C) 5.25% but less than 5.50%
    (D) 5.50% but less than 5.75%
    (E) 5.75% or more

31. On August 31, 1998, Smith will make a donation to the benefactor fund of his alma mater to provide for the following:

    1. A single four-year tuition scholarship
        Frequency and amount of tuition payments: Semiannually on each 9/1 and 3/1 in equal amounts
        Annual tuition for the 1998-1999 school year: $20,000
        Increase in annual tuition: 2.5% per year, compounded annually
        Date of first tuition payment from scholarship: 9/1/2001

    2. An annual perpetuity to the school
        Date of first perpetuity payment: 9/1/2005
        Amount of first perpetuity payment: $100,000
        Increase in annual perpetuity payments: 2.5% per year, compounded annually

    Interest rate on benefactor fund: 8% per year, compounded annually

    In what range is the amount of the donation?

    (A) Less than $1,202,000
    (B) $1,202,000 but less than $1,204,000
    (C) $1,204,000 but less than $1,206,000
    (D) $1,206,000 but less than $1,208,000
    (E) $1,208,000 or more

32. Terms of a 25-year annuity certain:
    Date of first payment: 12/31/99
    Frequency of payments: Annual
    Amount of each payment:
        First 10 years: $1,000 per year
        Next 10 years: $1,500 per year
        Final 5 years: $2,000 per year
    Interest rate: 8% per year, compounded semiannually

    In what range is the present value of the annuity as of 1/1/99?

    (A) Less than $12,500
    (B) $12,500 but less than $13,000
    (C) $13,000 but less than $13,500
    (D) $13,500 but less than $14,000
    (E) $14,000 or more

33. Terms of two actuarially equivalent annuities:
    Annuity A: $500 at the end of each of the first 3 months, and $1,000 at the end of each of the next 9 months.
    Annuity B: $$P$ at the end of each of the first 2 quarters, and $$2P$ at the end of each of the next 2 quarters.
    Interest rate: 8% per year, compounded monthly.

    In what range is $$P$?

    (A) Less than $1,770
    (B) $1,770 but less than $1,800
    (C) $1,800 but less than $1,830
    (D) $1,830 but less than $1,860
    (E) $1,860 or more

34. Terms of a perpetuity:
    Effective date: 1/1/99
    Frequency of payments: Annual
    Date of first payment: 12/31/99
    Amount of each payment: $$2X$
    Present value of future payments as of 1/1/2009: $$P$
    Terms of a 10-year annuity certain:
    Effective date: 1/l/99
    Frequency of payments: Annual
    Date of first payment: 12/31/99
    Amount of each payment: $$X$
    Accumulated value of payments as of 1/1/2009: $$P/2$
    Interest rate: i%

    In what range is i%?

    (A) Less than 7.25%
    (B) 7.25% but less than 7.75%
    (C) 7.75% but less than 8.25%
    (D) 8.25% but less than 8.75%
    (E) 8.75% or more

35. Fund balance as of 1/1/99: $12,000
Deposits to the fund: $100 on the last day of each month for 5 years
First deposit: 1/31/99
Withdrawals from the fund: $1,000 on the first day of each quarter
First withdrawal: 1/1/2006
No other deposits or withdrawals are made.
Interest rate: 8% per year, compounded monthly

In what range is the fund balance as of 12/31/2010?

(A) Less than $13,500
(B) $13,500 but less than $15,000
(C) $15,000 but less than $16,500
(D) $16,500 but less than $18,000
(E) $18,000 or more

36. Purchase date of a perpetuity: 1/1/99
Date of first payment: 3/31/99
Frequency of payments: Quarterly
Quarterly payments during each year as follows:

Quarter	Amount
1	$100
2	200
3	300
4	400

Interest rate: 10% per year, compounded annually.

In what range is the purchase price of the perpetuity?

(A) Less than $10,125
(B) $10,125 but less than $10,250
(C) $10,250 but less than $10,375
(D) $10,375 but less than $10,500
(E) $10,500 or more

37. Purchase date of a perpetuity-due: 1/1/2000
Level payment amount: $100
Frequency of payments: Annual
Cost of perpetuity: $1100
Interest rate of perpetuity: i%, compounded annually
Immediately following the payment on 1/1/2014, the remaining future payments are sold at a yield rate of i%. The proceeds are used to purchase an annuity certain as follows:
Term of annuity: 10 years
First payment of annuity: 1/1/2018
Frequency of annuity payments: Semi-annual on January 1 and July 1
Interest rate for annuity: 1/2 i% compounded annually

In what range is the semi-annual annuity payment:

(A) Less than $75
(B) $75 but less than $77
(C) $77 but less than $79
(C) $79 but less than $81
(E) $81 or more

38. Terms of a 20-year annuity-certain:

    Initial payment: \$300 due 1/1/2000

    Payment pattern:

    a) All payments are made on January 1
    b) Payments increase by \$300 each year beginning 1/1/2001 through 1/1/2009
    c) Payments decrease by \$200 each year beginning 1/1/2010 through 1/1/2019

    Interest rate: 7% per year, compounded annally for the first 10 years:
    6% per year, compounded annually thereafter.

    In what range is the present value of the annuity as of January 1, 2000?

    (A) Less than \$18,600
    (B) \$18,600 but less than \$18,800
    (C) \$18,800 but less than \$19,000
    (D) \$19,000 but less than \$19,200
    (E) \$19,200 or more

SOLUTIONS

S-5. Note that $s_{\overline{3n|}} = s_{\overline{2n|}}(1+i)^n + s_{\overline{n|}}$. Then $\frac{s_{\overline{3n|}}}{s_{\overline{2n|}}} = (1+i)^n + \frac{s_{\overline{n|}}}{s_{\overline{2n|}}} = \frac{105.2974}{43.7840} = 2.40493$. Therefore $s_{\overline{n|}} = [2.40493 - (1+i)^n]s_{\overline{2n|}} = (2.40493 - 2.0803)(43.7840) = 14.2136$, ANSWER C.

S-6. As of 1/1/88 the present value is

$$\begin{aligned} PV &= 10,000\left[\frac{1}{1.06} + \frac{1.03}{(1.06)^2} + \frac{(1.03)^2}{(1.06)^3} + \cdots + \frac{(1.03)^9}{(1.06)^{10}} + \frac{(1.03)^9}{(1.06)^{11}} + \cdots + \frac{(1.03)^9}{(1.06)^{15}}\right] \\ &= \frac{10,000}{1.06}\left[1 + \frac{1.03}{1.06} + \cdots + \left(\frac{1.03}{1.06}\right)^9\right] + \frac{10,000(1.03)^9}{(1.06)^{10}}\left[\frac{1}{1.06} + \cdots + \frac{1}{(1.06)^5}\right] \\ &= \frac{10,000}{1.06}\left[\frac{1 - \left(\frac{1.03}{1.06}\right)^{10}}{1 - \left(\frac{1.03}{1.06}\right)}\right] + \frac{10,000(1.03)^9 \cdot a_{\overline{5|}.06}}{(1.06)^{10}} \\ &= \frac{10,000}{1.06}\left(\frac{.24956}{.02830}\right) + \frac{10,000(1.30477)(4.21241)}{1.79085} \\ &= 83,193.55 + 30,690.56 = 113,884.11, \quad \text{ANSWER A.} \end{aligned}$$

S-7. On 1/1/88 the retirement benefit is a perpetuity-due, with present value $1000\ddot{a}_{\overline{10|}.08}$ for the first 10 years, and $v_{.08}^{10} \cdot \frac{1000}{d_{.06}}$ for the rest of the perpetuity. Then

$$PV = 1000(1.08)a_{\overline{10|}.08} + \frac{1000}{\frac{.06}{1.06}} \cdot v_{.08}^{10} =$$
$$1000(1.08)(6.71007) + (17,666.67)(.46319) = 15,429.90, \text{ ANSWER B.}$$

S-8. The initial deposit earns 8% each year, or 8000 per year. These additions to the contract earn 6%, with the first addition reinvested at the end of the first year. Thus the 8000 deposits accumulate to $8000s_{\overline{10|}.06} = 105,446.67$. Along with the initial deposit itself we have an accumulated value on 1/1/98 of $205,446.67$, ANSWER C.

S-9. Since the interest earned by Account A, in amount of $(.06)(200,000) = 12,000$ is invested in Account B, then the value of Account A remains level at $200,000$. Account B receives deposits of $12,000$ at the end of each of 1989 through 1993, so the balance in Account B on 1/1/94 is $100,000(1+i)^5 + 12,000s_{\overline{5|}i}$, where i is the effective *annual* rate earned by Account B. Note that i is given by $1+i = (1.045)^2$, so $i = .092025$, and $s_{\overline{5|}i} = 6.00891$. Thus we have a total value on 1/1/94 of $200,000 + 100,000(1.55297) + 12,000(6.00891) = 427,403.92$, ANSWER D.

10. The participant is aged $39\frac{1}{2}$ when the plan is established on 1/1/89, and retires on 7/1/2014 at age 65. No allocation is made in year 2014.

The first allocation is made on 12/31/89, in amount of $(.10)(1.05)(40,000)$, and accumulated to 7/1/2014 at 7%, giving $(.10)(1.05)(40,000)(1.07)^{24.5}$.

The projected allocation on 12/31/90 will be $(.10)(1.05)^2(40,000)$, and accumulates to $(.10)(1.05)^2(40,000)(1.07)^{23.5}$ on 7/1/2014.

The final projected allocation on 12/31/2013 will be $(.10)(1.05)^{25}(40,000)$, and will accumulate to $(.10)(1.05)^{25}(40,000)(1.07)^{.5}$ on 7/1/2014.

The total projected accumulation is therefore

$$(.10)(40,000)(1.05)(1.07)^{24.5}\left[1+\frac{1.05}{1.07}+\left(\frac{1.05}{1.07}\right)^2+\cdots+\left(\frac{1.05}{1.07}\right)^{24}\right]$$

$$= 22,036.98\left(\frac{1-\left(\frac{1.05}{1.07}\right)^{25}}{1-\frac{1.05}{1.07}}\right) = 443,374.85, \quad \text{ANSWER D.}$$

11. $PV_A = 1000a_{\overline{n|}} \qquad PV_B = 2000(a_{\overline{m+2n|}} - a_{\overline{m|}}) \qquad PV_C = P\cdot a_{\overline{m+n|}} = PV_A + PV_B$

We need to solve for $a_{\overline{m+2n|}}$. We have $a_{\overline{m+2n|}} = a_{\overline{n|}} + v^n\cdot a_{\overline{m+n|}}$, and we know that $v^n = 1 - ia_{\overline{n|}} = 1 - 11.101i$. Similarly $v^m = 1 - ia_{\overline{m|}} = 1 - 8.273i$, and $v^{m+n} = 1 - ia_{\overline{m+n|}} = 1 - 12.780i$. But $v^{m+n} = v^m\cdot v^n$, so $1 - 12.780i = (1-11.101i)(1-8.273i)$, leading to $1 - 12.780i = 1 - 19.374i + 91.838573i^2$. This quadratic solves for $i = .0718$, giving us $v^n = 1 - 11.101(.0718) = .20294$. Finally,

$$P = \frac{PV_A + PV_B}{a_{\overline{m+n|}}} = \frac{1000(11.101) + 2000[11.101 + (.20294)(12.780) - 8.237]}{12.780} = 1722.72,$$

ANSWER C.

12. $\ddot{a}_{\overline{t+1|}} = 1 + v\cdot\ddot{a}_{\overline{t|}}$, so $v = \dfrac{\ddot{a}_{\overline{t+1|}} - 1}{\ddot{a}_{\overline{t|}}} = \dfrac{6.950}{7.452} = .93263$. Then $d = 1 - v = .06737$ and $1 + i = \frac{1}{v} = 1.07223$. Then $\ddot{s}_{\overline{25|}} = \dfrac{(1+i)^{25}-1}{d} = \dfrac{(1.07223)^{25}-1}{.06737} = 70.02271$, ANSWER D.

13. The quarterly payments are level within each year, and then increase annually. The effective quarterly interest rate is $j = (1.10)^{1/4} - 1 = .02411$. The end-of-year equivalent of 1 per quarter is $X = s_{\overline{4|}j} = 4.14766$. The end-of-year equivalents in future years are $X(1.08)$, $X(1.08)^2$, and so on. Then the present value, at rate $i = .10$, is

$$PV = \frac{X}{1.10} + \frac{X(1.08)}{(1.10)^2} + \frac{X(1.08)^2}{(1.10)^3} + \cdots$$

$$= \frac{4.14766}{1.10}\left[1 + \left(\frac{1.08}{1.10}\right) + \left(\frac{1.08}{1.10}\right)^2 + \cdots\right]$$

$$= \frac{4.14766}{1.10}\left[\frac{1}{1-\left(\frac{1.08}{1.10}\right)}\right] = 207.38, \quad \text{ANSWER B.}$$

S-14. Let the first quarterly payment be Q. Then the 4/1/96 payment is $[Q-(20)(300)]$, since there have been 20 quarterly decreases of 300 each since 4/1/91. As of 1/1/96 there are 20 payments remaining, with present value

$$50{,}000 = [Q-(19)(300)]a_{\overline{20}|.02} - 300(Ia)_{\overline{20}|.02},$$

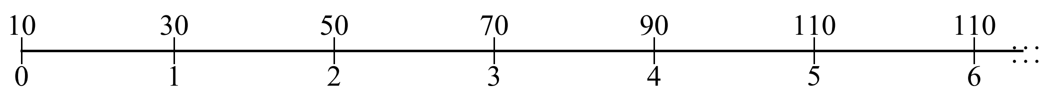
so that

$$Q = \frac{50,000 + 300(Ia)_{\overline{20}|.02} + 5700a_{\overline{20}|.02}}{a_{\overline{20}|.02}} = 11{,}710.88, \text{ ANSWER C.}$$

S-15. There are 40 quarterly payments in the pattern $1,2,3,4,1,2,3,4,\ \ldots,$ with payments made at the ends of quarters. Note that there are 10 annual payments at each payment amount. The present value of the \$1 payments is $\ddot{a}_{\overline{10}|}$ as of 3/31/92; the present value of the \$2 payments is $2\ddot{a}_{\overline{10}|}$ as of 6/30/92, and so on. Then the overall present value as of 1/1/92 is

$$\begin{aligned} PV &= \ddot{a}_{\overline{10}|} \cdot v^{1/4} + 2\ddot{a}_{\overline{10}|} \cdot v^{1/2} + 3\ddot{a}_{\overline{10}|} \cdot v^{3/4} + 4\ddot{a}_{\overline{10}|} \cdot v \\ &= \ddot{a}_{\overline{10}|}\left(v^{1/4}+2v^{1/2}+3v^{3/4}+4v\right) \\ &= 7.51523\,[.98322+2(.96673)+3(.95052)+4(.93457)] \\ &= 71.44, \text{ ANSWER D.} \end{aligned}$$

S-16. The payment schedule is a follows:



Recall that the present value of the level perpetuity is $\frac{110}{.08} = 1375$ as of time $t=4$. Then the entire present value as of time $t=0$ is

$$\begin{aligned} PV &= 10 + \frac{30}{1.08} + \frac{50}{(1.08)^2} + \frac{70}{(1.08)^3} + \frac{90+1375}{(1.08)^4} \\ &= 10 + 27.78 + 42.87 + 55.57 + 1076.82 \\ &= 1213.04, \text{ ANSWER C.} \end{aligned}$$

S-17. The interest rate is $i=.04$ effective per half-year, which is $j=(1.04)^2-1=.0816$ effective per year. The present value of the annuity certain is

$$PV_A = 50{,}000\ddot{a}_{\overline{20}|j} = 50{,}000(1.0816)\left(\frac{1-(1.0816)^{-20}}{.0816}\right) = 524{,}702.54\,.$$

The equivalent effective monthly rate is $j' = (1.04)^{1/6}-1 = .00656$. The present value of the perpetuity is

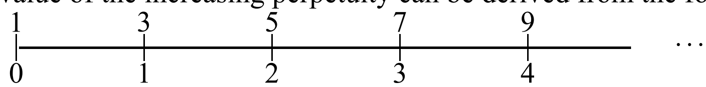
$$PV_P = X(1.00656)\left(\frac{1}{.00656}\right) = 153.44X = 524{,}702.54,$$

so $X = \dfrac{524{,}702.54}{153.44} = 3419.59$, ANSWER B.

S-18. The present value of the increasing perpetuity is

$$\begin{aligned} PV &= 500v + 500(1.05)v^2 + 500(1.05)^2v^3 + \cdots \\ &= \frac{500}{1.08}\left(1 + \frac{1.05}{1.08} + \left(\frac{1.05}{1.08}\right)^2 + \cdots\right) \\ &= \frac{500}{1.08}\left(\frac{1}{1 - \frac{1.05}{1.08}}\right) = 16{,}666.67, \text{ ANSWER D.} \end{aligned}$$

S-19. The present value of the increasing perpetuity can be derived from the following diagram:

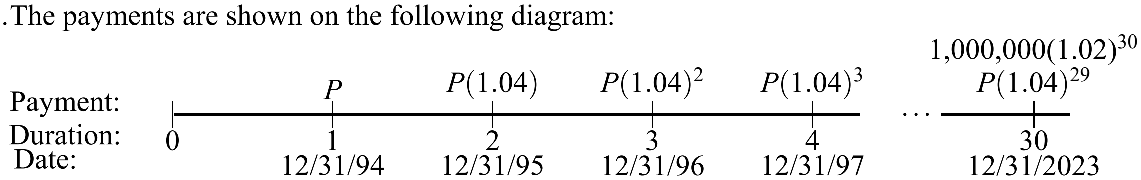


A level perpetuity-due with payments of 1 each year has present value $\frac{1}{d}$ at $t = 0$. A level perpetuity of 2 each year, starting at $t = 1$, has present value $\frac{2}{d}$ at $t = 1$, and value $\frac{2}{d} \cdot v$ at $t = 0$. Another level perpetuity of 2 each year, starting at $t = 2$, has present value $\frac{2}{d}$ at $t = 2$, and value $\frac{2}{d} \cdot v^2$ at $t = 0$. Continuing in this manner the total present value is seen to be

$$PV = \frac{1}{d} + \frac{2}{d}(v + v^2 + \cdots) = \frac{1}{d} + \frac{2}{d}\left(\frac{1}{i}\right) = \frac{25}{d}.$$

Multiplying by d and solving for i we find $i = \frac{2}{24}$, ANSWER C.

S-20. The payments are shown on the following diagram:



The effective annual rate of interest is $j = (1.05)^2 - 1 = .1025$, so the present value of all payments is

$$\begin{aligned} 1{,}000{,}000 &= P \cdot v_i + P(1.04) \cdot v_i^2 + P(1.04)^2 \cdot v_i^3 + \cdots + P(1.04)^{29} \cdot v_i^{30} + 1{,}000{,}000(1.02)^{30} \cdot v_i^{30} \\ &= \frac{P}{1.1025}\left[1 + \left(\frac{1.04}{1.1025}\right) + \left(\frac{1.04}{1.1025}\right)^2 + \cdots + \left(\frac{1.04}{1.1025)}\right)^{29}\right] + 1{,}000{,}000\left(\frac{1.02}{1.1025}\right)^{30} \\ &= \frac{P}{1.1025}(14.57704) + 96{,}972.20. \end{aligned}$$

The last equation solves for $P = 68{,}298.38$, so the total payment on 12/31/2023 is $68{,}298.38(1.04)^{29} + 1{,}000{,}000(1.02)^{30} = 2{,}024{,}358.70$, ANSWER D.

S-21. Recall that a unit increasing perpetuity-due of 1, 2, 3, … has present value of $\frac{1}{d^2}$. Then as of 1/1/94 this increasing perpetuity-due with payments of 10, 20, 30, … has present value of $320{,}000 = \frac{10}{d^2}$, from which we find $d = \sqrt{\frac{10}{320{,}000}} = .00559$. Note that this d is effective monthly. The corresponding effective monthly rate of interest is $j = \frac{d}{1-d} = .00562$, and the corresponding effective annual rate of interest is $i = (1+j)^{12} - 1 = .06958$, ANSWER D.

S-22. The present value is given by

$$PV = 1500a_{\overline{n|}} + 2500v^n \cdot a_{\overline{m|}} + 3500v^{n+m} \cdot a_{\overline{2n|}} \,.$$

We are given $a_{\overline{m|}} = \frac{1-v^m}{i} = \frac{1-.215}{i} = 9.818$, from which we find $i = .08$. Then from $a_{\overline{n|}} = \frac{1-v^n}{.08} = 8.559$ we find $v^n = .31528$. Next we can find $a_{\overline{2n|}} = \frac{1-v^{2n}}{i} = \frac{1-(.31528)^2}{.08} = 11.257$. Finally we have $PV = 1500(8.559) + 2500(.31528)(9.818) + 3500(.31528)(.215)(11.257) = 23{,}247.87$, ANSWER E.

S-23. The payment pattern for the annuity is as follows:

Time	0	1	2	3	4	5	6	7	8	9	10	11
Payment		100	200	400	800	1600	3200	6400	12,800	25,600	51,200	51,200…

The present value of the first payment is $\frac{100}{1.07}$. The present value of the second payment is $\frac{200}{(1.07)^2} = \frac{2}{1.07}\left(\frac{100}{1.07}\right)$. The present value of the third payment is $\frac{400}{(1.07)^3} = \frac{2}{1.07}\left(\frac{2}{1.07}\right)\left(\frac{100}{1.07}\right) = \frac{100}{1.07}\left(\frac{2}{1.07}\right)^2$. In similar manner, the present value of the tenth payment is $\frac{100}{1.07}\left(\frac{2}{1.07}\right)^9$.

Then the present value of the first ten payments is

$$PV_{1-10} = \frac{100}{1.07}\left[1 + \frac{2}{1.07} + \left(\frac{2}{1.07}\right)^2 + \cdots + \left(\frac{2}{1.07}\right)^9\right] = \frac{100}{1.07}\left[\frac{1-\left(\frac{2}{1.07}\right)^{10}}{1-\frac{2}{1.07}}\right] = 55{,}865.60.$$

The present value of all payments after the first ten is $PV_{11-\infty} = \frac{51{,}200(1.07)^{-10}}{.07} = 371{,}821.40$. Then the total present value is $PV = 55{,}865.60 + 371{,}821.40 = 427{,}687.00$, ANSWER B.

S-24. The monthly interest rate for the payment plan is $j = .005$, so the monthly payment is

$P = \frac{20{,}000 - 500}{a_{\overline{60|}.005}} = 376.99$. The two options are equivalent at the monthly rate $j' = .0075$, so we have the equation of value

$$20{,}000 - X = 500 + 376.99 a_{\overline{60|}.0075},$$

from which we find $X = 19{,}500 - 376.99 a_{\overline{60|}.0075} = 1339.14$, ANSWER A.

S-25. The payment schedule is shown on the following diagram:

	N	$2N$	N	N	$2N$	N	N	$2N$	N	N	$2N$	N	N	$2N$	N
0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15

At time $t = 0$ on 1/1/94, the present value of all future payments is

$$\begin{aligned} P &= N \cdot a_{\overline{15|}} + N \cdot v^2 \left[1+v^3+v^6+v^9+v^{12}\right] \\ &= N\left[a_{\overline{15|}} + v^2\left(\frac{1-v^{15}}{1-v^3}\right)\right] \\ &= 12.13929N. \end{aligned}$$

At time $t = 3$ on 1/1/97 (just before the 1/1/97 payment), the retrospective outstanding balance is

$$P(1.07)^3 - N(1.07)^2 - 2N(1.07) = P - 11{,}061$$

since the retrospective OB must equal the prospective OB given by the present value of then future payments. This equation gives us

$$.22504P = 3.2849N - 11{,}061.$$

Substituting $P = 12.13929N$ from above we have

$$3.2849N - .22504(12.13929N) = 11{,}061$$

or

$$N = \frac{11{,}061}{.55307} = 19{,}999.13, \text{ ANSWER E.}$$

S-26. The payment at time t is $P_t = \frac{(t+1)(t+2)}{2}$, so the present value of the perpetuity is

$$PV = \sum_{t=0}^{\infty} P_t \cdot v^t = \sum_{t=0}^{\infty} \frac{1}{2}(t^2+3t+2)v^t$$

$$= \frac{1}{2}\sum_{t=1}^{\infty} t^2 \cdot v^t + \frac{3}{2}\sum_{t=1}^{\infty} t \cdot v^t + \sum_{t=0}^{\infty} v^t.$$

The values of $\sum_{t=0}^{\infty} v^t = \ddot{a}_{\overline{\infty|}} = \frac{1+i}{i}$ and $\sum_{t=1}^{\infty} t \cdot v^t = (Ia)_{\overline{\infty|}} = \frac{1}{i} + \frac{1}{i^2}$ are easily found.

The first summation can be evaluated as follows:

Let $$A = v + 4v^2 + 9v^3 + 16v^4 + 25v^5 + 36v^6 + \cdots$$

$$(1+i)A = 1 + 4v + 9v^2 + 16v^3 + 25v^4 + 36v^5 + \cdots.$$

Subtracting we find

$$iA = 1 + 3v + 5v^2 + 7v^3 + 9v^4 + 11v^5 + \cdots$$
$$= \ddot{a}_{\overline{\infty|}} + 2 \cdot (Ia)_{\overline{\infty|}},$$

so $A = \frac{1}{i}\left[\frac{1+i}{i} + 2\left(\frac{1}{i} + \frac{1}{i^2}\right)\right]$.

Putting the pieces together we have

$$PV = \frac{1}{2} \cdot \frac{1}{i}\left[\frac{1+i}{i} + 2\left(\frac{1}{i} + \frac{1}{i^2}\right)\right] + \frac{3}{2}\left(\frac{1}{i} + \frac{1}{i^2}\right) + \frac{1+i}{i}$$

$$= \left(\frac{1}{2}\right)(4)[5 + 2(4+16)] + \frac{3}{2}(4+16) + 5 = 125, \text{ ANSWER D.}$$

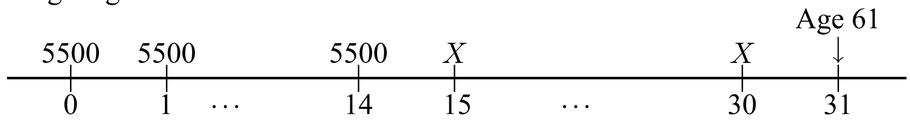
S-27. The payments are made monthly, but the increase occurs annually. The value on 12/31/81 of the deposits made in 1981 is $25s_{\overline{12|}.01} = 317.075$. Then the value on 12/31/82 of the 1982 deposits is 317.075(1.12), the value on 12/31/83 of the 1983 deposits is $317.075(1.12)^2$, and so on. The value on 12/31/98 of the 1998 deposits is $317.075(1.12)^{17}$. The effective annual interest rate is $j = (1.10)^{12} - 1$. The total accumulated value on 1/1/99 is therefore

$$AV = 317.075\left[(1+j)^{17} + (1.12)(1+j)^{16} + (1.12)^2(1+j)^{15} + \cdots + (1.12)^{17}\right]$$

$$= 317.075(1+j)^{17}\left[1 + \frac{1.12}{1+j} + \left(\frac{1.12}{1+j}\right)^2 + \cdots + \left(\frac{1.12}{1+j}\right)^{17}\right]$$

$$= 317.075[(1.10)^{12}]^{17}\left[\frac{1 - \left(\frac{1.12}{(1.01)^{12}}\right)^{18}}{1 - \frac{1.12}{(1.01)^{12}}}\right] = 41{,}285.71, \text{ ANSWER D.}$$

S-28. Let age 30 be time 0, age 61 be time 31, and age 45 be time 15. The question is represented on the following diagram:



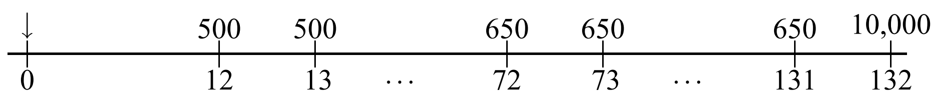
We seek the value of X so that the 15 deposits of size 5500 plus the 16 deposits of size X will accumulate to 1,000,000 at $t = 31$ at 9% effective annual rate. Thus we have

$$5500 s_{\overline{15}|.09}(1.09)^{17} + X\ddot{s}_{\overline{16}|} = 1{,}000{,}000$$

which solves for

$$X = \frac{1{,}000{,}000 - 5500(29.36089)(4.32763)}{35.97375} = 8371.47, \text{ ANSWER C.}$$

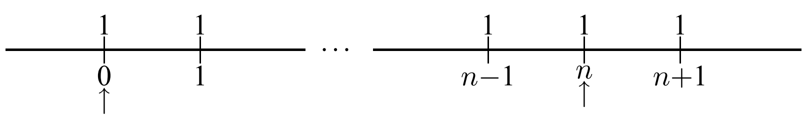
S-29. Let 1/1/98 be time $t = 0$, so that 1/1/99 is $t = 12$ and 1/1/09 is $t = 132$, where time is measured in months. The question is represented on the following diagram:



The effective monthly interest rate is $j = (1.07)^{1/12} - 1$. The value of the 60 payments of 500 each, valued at $t = 12$, is $500\ddot{a}_{\overline{60}|j} = 25{,}527.04$. The value of the 60 payments of 650 each, valued at $t = 72$, is $650\ddot{a}_{\overline{60}|j} = 33{,}185.16$. Then the total present value at $t = 0$ of all payments is

$$PV = 25{,}527.04(1.07)^{-1} + 33{,}185.16(1.07)^{-6} + 10{,}000(1.07)^{-11} = 50{,}720.66, \text{ ANSWER B.}$$

S-30. First we observe that $\ddot{a}_{\overline{n+2}|}$ is the value at time $t = 0$ of the following sequence of $n+2$ payments:



If valued at time $t = n$, the same sequence has the value $\ddot{a}_{\overline{n+2}|} \cdot (1+i)^n$. Another way to express the value of the $n+2$ payments at time $t = n$ is $\ddot{s}_{\overline{n}|} + 1 + v$. Thus we have the equation

$$\ddot{a}_{\overline{n+2}|} \cdot (1+i)^n = \ddot{s}_{\overline{n}|} + 1 + 1 - d,$$

since $v = 1 - d$. But

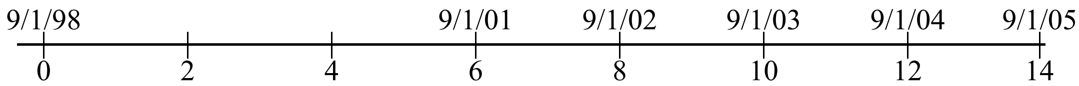
$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = 52.344,$$

so we find $(1+i)^n = 1 + 52.344d$. Then we have

$$(14.030)(1 + 52.344d) = 52.344 + 2 - d,$$

from which we find $d = .05482$, and $i = \dfrac{d}{1-d} = .05800$, ANSWER E.

S-31. For the scholarship, the important dates are shown on the following diagram, where time is measured in half-years:



The tuition is 20,000$(1.025)^3$ for school year 2001-02, 20,000$(1.025)^4$ for school year 2002-03, 20,000$(1.025)^5$ for school year 2003-04, and 20,000$(1.025)^6$ for school year 2004-05. Note that the tuition *increases* annually but the tuition *payments* are made semiannually on 9/1 and 3/1, in equal amounts. Then the present value of the tuition payments is

$$PV_1 = 10{,}000[(1.025)^3(v^6+v^7) + (1.025)^4(v^8+v^9) + (1.025)^5(v^{10}+v^{11}) + (1.025)^6(v^{12}+v^{13})],$$

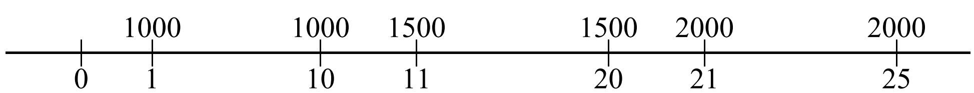
where the v^t values are evaluated at $j = (1.08)^{1/2} - 1$. Evaluating we find $PV_1 = 62{,}144.96$.

For the perpetuity, time can be measured in years. The first payment of 100,000 is made at time $t = 7$, the second payment of 100,000(1.025) is made at time $t = 8$, and so on. The present value of the perpetuity is

$$\begin{aligned} PV_2 &= 100{,}000[(1.08)^{-7} + (1.025)(1.08)^{-8} + (1.025)^2(1.08)^{-9} + \cdots] \\ &= \frac{100{,}000}{(1.08)^7}\left[1 + \frac{1.025}{1.08} + \left(\frac{1.025}{1.08}\right)^2 + \cdots\right] \\ &= \frac{100{,}000}{(1.08)^7}\left(\frac{1}{1-\frac{1.025}{1.08}}\right) = 1{,}145{,}765.70. \end{aligned}$$

Then the total donation is $PV_1 + PV_2 = 1{,}207{,}910.70$, ANSWER D.

S-32. The payments are shown on the following diagram:



The effective annual interest rate is $j = (1.04)^2 - 1 = .0816$. Then the present value is

$$\begin{aligned} PV &= 1000a_{\overline{25}|j} + 500a_{\overline{15}|j}v_j^{10} + 500a_{\overline{5}|j}v_j^{20} \\ &= 1000(10.53048) + 500(8.47649)(.45639) + 500(3.97591)(.20828) \\ &= 12{,}878.82, \text{ ANSWER B.} \end{aligned}$$

S-33. The interest rate is $j = 2/3\%$ effective per month. For Annuity A,

$$PV_A = 500a_{\overline{3}|j} + 1000a_{\overline{9}|j}v_j^3 = 500(2.95992) + 1000(8.70784)(.98026) = 10{,}015.97.$$

For Annuity B, the effective quarterly interest rate is $k = (1.0066\dot{6})^3 - 1 = .02013$. Then

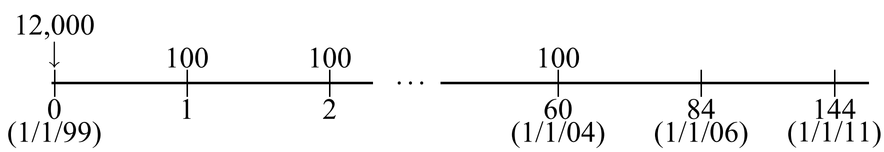
$$PV_B = P[v_k + v_k^2 + 2v_k^3 + 2v_k^4] = 5.67186P.$$

Equating $PV_A = PV_B$, we find

$$P = \frac{10{,}015.97}{5.67186} = 1765.90, \text{ ANSWER A.}$$

S-34. The present value of the perpetuity is $P = \frac{2X}{i}$. The accumulated value of the annuity-certain is $\frac{P}{2} = X\left(\frac{(1+i)^{10} - 1}{i}\right)$, so $P = \frac{2X}{i}((1+i)^{10} - 1)$. Equating the two expressions for P we find $(1+i)^{10} = 2$, so $i = 2^{1/10} - 1 = .07177$, ANSWER A.

S-35. The details of the fund are shown on the following diagram:



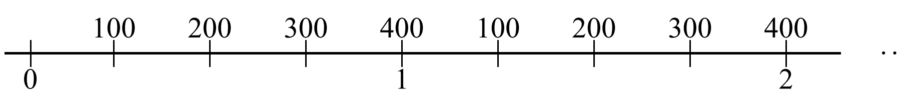
The effective monthly interest rate on the fund is $j = .0066\dot{6}$. Deposits are made for 60 months, and the fund balance then accumulates for 24 months more. Then the fund balance on 1/1/06 is

$$12{,}000(1+j)^{84} - 100s_{\overline{60}|j}(1+j)^{24} = 29{,}586.93.$$

At this point quarterly withdrawals of 1000 begin, and continue for 20 quarters. The effective quarterly interest rate is $k = (1+j)^3 - 1 = .02013$. Finally, the fund balance on 12/31/10 is

$$29{,}586.93(1+k)^{20} - 1000\ddot{s}_{\overline{20}|k} = 19{,}258.23, \text{ ANSWER E.}$$

S-36. The payments are shown on the following diagram:



The effective quarterly interest rate is $j = (1.10)^{1/4} - 1 = .024114$. The four payments within each year have equivalent value of $100(Is)_{\overline{4}|j} = 1024.39$ as of the end of each year. Then the present value of the perpetuity is

$$PV = \frac{1024.39}{.10} = 10{,}243.94, \text{ ANSWER B.}$$

S-37. The original present value of the perpetuity-due is

$$PV_0 = 1100 = \frac{100}{d} = \frac{100}{i/1+i},$$

from which we find $i =$ .10. When the future payments are sold on 1/1/14 to yield rate i, the sale price is the then present value of future payments, which is

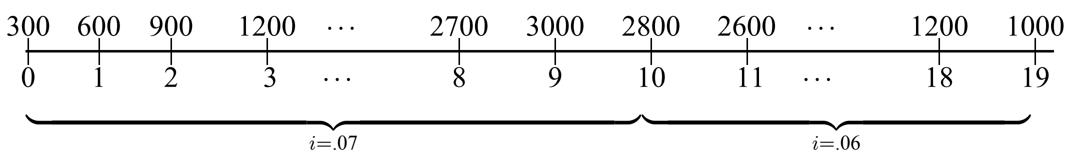
$$PV_F = \frac{100}{i} = 1000,$$

since the future payments then form a perpetuity-immediate. The semi-annual annuity is valued at rate $\frac{1}{2}i = .05$ effective annual, or $j = (1.05)^{1/2} - 1 = .024695$ effective semiannual. The first annuity payment is on 1/1/18. At that date the present value of the annuity is

$$PV = P \cdot \ddot{a}_{\overline{20}|j} = 16.02018P,$$

and its present value on 1/1/14 is $16.02018P(1.05)^{-4} = 1000$, from which we find $P = 75.87$, ANSWER B.

S-38. The payments and effective interest rates are shown on the following diagram, where 1/1/00 is at $t = 0$



The value at $t = 10$ of the payments at $t = 10$ thru $t = 19$ is

$$PV_{10} = 800\ddot{a}_{\overline{10}|.06} + 200(D\ddot{a})_{\overline{10}|.06} = 800(7.80171) + 200(46.63825) = 15{,}569.02.$$

The value at $t = 0$ of the payments at $t = 0$ thru $t = 9$ is

$$PV_0 = 300(I\ddot{a})_{\overline{10}|.07} = 300(37.17074) = 11{,}151.22.$$

The overall present value is

$$PV = PV_0 + PV_{10}(1.07)^{-10} = 19{,}065.73, \text{ ANSWER D.}$$

Unit I-A3
Fund Yield Rates

Overview Commentary

All financial transactions involve an interest rate, which is sometimes referred to as the *yield rate* or the *rate of return* on the transaction. In the basic case of X invested at time 0 accumulating to Y at time n, we have the simple equation of value $Y = X(1+i)^n$, where i denotes the yield rate on the transaction. In this transaction, X can be viewed as a deposit into an investment fund at $t = 0$ and Y as a withdrawal from the fund at $t = n$.

Most financial transactions are more complicated than this basic case, in that they involve multiple deposits into and withdrawals from an investment fund. In the more complicated case, a closed-form solution for the yield rate does not exist and we resort to various techniques to approximate the yield rate. Two such techniques, an understanding of which is required for the EA-1 exam, are the *dollar-weighted method* and the *time-weighted method*. A description of these approximate methods is found in Section 5.2 of Broverman, Sections 2.2 and 2.3 of Parmenter, and Sections 5.5 and 5.6 of Kellison. SOA Study Note E1-61-91, entitled “Measurement of Investment Return,” also discusses these two approximate methods, as well as presenting an historical review of other investment return measurement techniques.

The following sample questions illustrate the inclusion of the topic of investment rate of return on the EA-1 exam.

SAMPLE QUESTIONS

39.

	Date	Amount
Trust Market Values:	1/1/88	$75,000
	3/31/88	80,800
	6/30/88	93,700
	9/30/88	73,200
	1/1/89	77,600
Contributions Received:	4/1/88	15,000
	12/31/88	2,700
		17,700
Benefits Paid:	7/1/88	12,200
	9/29/88	10,100
		22,300

In what range is the time-weighted rate of return for 1988?

(A) Less than 9.25%
(B) 9.25% but less than 9.50%
(C) 9.50% but less than 9.75%
(D) 9.75% but less than 10.00%
(E) 10.00% or more

40.

	Date	Amount
Trust market values:	1/1/89	$100,000
	4/1/89	130,000
	7/1/89	95,000
	10/1/89	130,000
	1/1/90	150,000
Contributions received:	3/31/89	20,000
	6/30/89	10,000
	9/30/89	20,000
	12/31/89	10,000
Benefits paid:	6/30/89	40,000
	12/31/89	5,000

In what range is the time-weighted rate of return for 1989?

(A) Less than 34.0%
(B) 34.0% but less than 35.0%
(C) 35.0% but less than 36.0%
(D) 36.0% but less than 37.0%
(E) 37.0% or more

41. Value of assets in trust fund:

1/1/90	100,000
4/1/90	160,000
10/1/90	130,000
12/31/90	130,000

Contributions for 1990: 50,000 paid on 3/31/90
Payments of 1990: 30,000 paid on 9/30/90

Consider the following methods for determining the fund's rate of return:

I. Time-weighted method
II. Uniform distribution throughout the year of all contributions and payments is assumed; simple interest is used
III. Dollar-weighted method; simple interest is used

Which, if any, of the following rankings reflects the relative magnitude of the calculated rate of return under each method?

A. I > II > III
B. I > III > II
C. II > I > III
D. II > III > I
E. The correct answer is not given by A, B, C, or D above.

42. Market value of fund:

	Date	Amount
Market value of fund:	1/1/91	200,000
	4/1/91	200,000
	7/1/91	286,000
	10/1/91	276,000
	1/1/92	260,000
Contributions to fund:	6/30/91	80,000
Benefit payments from fund	3/31/91	10,000
	6/30/91	10,000
	9/30/91	10,000
	12/31/91	10,000

Consider the following measures of the fund's rate of return for 1991:

I. Time-weighted rate of return
II. Dollar-weighted rate of return using simple interest
III. Annual rate of return based on an assumed uniform distribution of all contributions and benefit payments throughout the year using simple interest.

Which of the following shows the relative magnitude of these measures?

A. I > III > II
B. II = III, and I > II or III
C. II > III > I
D. III > I > II
E. The correct answer is not given by A, B, C, or D above

43. Value of assets in a trust fund:

As of 01/01/94:	$100,000
As of 04/01/94:	$150,000
As of 07/01/94:	$120,000
As of 10/01/94:	$200,000
As of 12/31/94:	$200,000

Contributions in 1994:

$10,000 paid on 3/31/94
$30,000 paid on 6/30/94
$50,000 paid on 9/30/94

Payments in 1994:

$10,000 paid on 3/31/94
$10,000 paid on 6/30/94
$10,000 paid on 9/30/94

In what range is the absolute value of the difference between the time-weighted rate of return on assets and the dollar-weighted rate of return on assets in 1994?

(A) Less than 2%
(B) 2% but less than 4%
(C) 4% but less than 6%
(D) 6% but less than 8%
(E) 8% or more

44. Market value of pension fund:

1/01/94	$100,000
4/01/94	100,000
7/01/94	150,000
10/01/94	140,000
1/01/95	130,000

Cash flow in fund:

Date	Contributions	Benefit Payments
3/31/94	--	$10,000
6/30/94	$50,000	10,000
9/30/94	--	10,000
12/31/94	--	10,000

Consider the following measures of the fund's rate of return for 1994:

I. Time-weighted rate of return.
II. Dollar-weighted rate of return using simple interest.
III. Effective rate derived by compounding the quarterly rates of return using simple interest during the quarter and assuming a uniform distribution of all cash flows throughout each quarter.

Which of the following is true?

(A) I > II > III
(B) I > III > II
(C) II > III > I
(D) III > II > I
(E) The correct answer is not given by A, B, C, or D

45. Market value of a pensions fund:

Date	Value
1/1/96	1,200,000
4/1/96	1,400,000
8/1/96	1,000,000
11/1/96	1,600,000
12/31/96	1,800,000

Cash flow in fund:

Date	Contributions	Benefit Payments
3/31/96	$100,000	$150,000
7/31/96	250,000	150,000
10/31/96	350,000	150,000

In what range is the absolute value of the difference between the dollar-weighted rate of return and the time-weighted rate of return?

(A) Less than 5.70%
(B) 5.70% but less than 5.80%
(C) 5.80% but less than 5.90%
(D) 5.90% but less than 6.00%
(E) 6.00% or more

46. Terms of an annuity:

Date of purchase: 1/1/96
Date of first payment: 1/1/96
Amount of each payment: $1000
Frequency of payments: Quarterly
Number of payments: 80
Interest rate: 5% per year, compounded quarterly

All annuity payments are reinvested by the purchaser for the remaining term of the annuity at 10% per year, compounded semiannually.

In what range is the total effective annual rate of return, compounded annually, to the purchaser of this arrangement?

(A) Less than 8.20%
(B) 8.20% but less than 8.25%
(C) 8.25% but less than 8.30%
(D) 8.30% but less than 8.35%
(E) 8.35% or more

47. Market value of a pension fund:

Date	Value Before Cash Flow
1/1/97	$1,000,000
7/1/97	1,030,000
X	1,025,000
12/31/97	1,150,000

Total cash flow in fund in 1997:

Date	Contributions	Benefit Payments
7/1/97	$ 0	$50,000
X	100,000	0

The time-weighted and dollar-weighted rates of return for the fund are equal.

In what range is *X*?

(A) Before 8/1/97
(B) 8/1/97 to 8/31/97
(C) 9/1/97 to 9/30/97
(D) 10/1/97 to 10/31/97
(E) 11/1/97 or after

48. Market value of a pension fund on 12/31/96: $60,000
Contribution made on 10/31/97: $30,000
Benefit payments made during 1997: $1,000 per month, paid on the first day of each month
Using simple interest:
Dollar-weighted rate of return for 1997: 7.0%
Expected rate of return for 1997: 6.0%

In what range is the investment gain for 1997?

(A) Less than $590
(B) $590 but less than $640
(C) $640 but less than $690
(D) $690 but less than $740
(E) $740 or more

49. Market value of a pension fund:

Date	Value
1/1/99	\$100,000
4/1/99	90,000
7/1/99	95,000
10/1/99	185,000
1/1/2000	180,000

Contributions and Benefit Payments:

Date	Contributions	Benefit Payments
3/31/99	\$0	\$20,000
9/30/99	75,000	0

Rates of return on fund:
Time weighted $= t$
Dollar weighted $= d$
Dollar weighted assuming cash flows at mid-year $= m$

Which of the following is true?

(A) $t > m > d$
(B) $t > d > m$
(C) $m > t > d$
(D) $d > t > m$
(E) None of the above

50. Market value of a pension fund on 12/31/98: \$1,750,000
Benefit payments are made on the first day of each calendar quarter.
Contributions are made on the last day of each calendar quarter and included in the ending balance.
Time-weighted rate of return for 1999: 8.79%
1999 fund data:

Quarter	Benefit Payments	Contributions	Quarterly Return	Ending Balance
1	\$22,000	\$30,000	1.02%	\$1,775,626
2	43,000	85,000	5.19%	1,907,549
3	27,994	y	−2.26%	1,937,033
4	39,228	27,617	j%	z

In what range is the dollar-weighted rate of return for 1999?

(A) Less than 8.68%
(B) 8.68% but less than 8.73%
(C) 8.73% but less than 8.78%
(D) 8.78% but less than 8.83%
(E) 8.83% or more

51. Market value of a pension fund:

Date	Value
1/1/2000	50,000
3/31/2000	60,000
6/30/2000	45,000
9/30/2000	40,000
12/31/2000	65,000

Contributions and benefit payments:

Date	Contributions	Benefit Payments
4/1/2000	0	P
7/1/2000	17,000	P
10/1/2000	55,000	P

Dollar-weighted rate of return for 2000 using simple interest: 7.00%

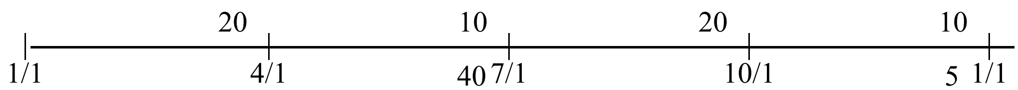
In what range is the time-weighted rate of return for 2000?

(A) Less than 6.00%
(B) 6.00% but less than 8.00%
(C) 8.00% but less than 10.00%
(D) 10.00% but less than 12.00%
(E) 12.00% or more

SOLUTIONS

S-39. For the first quarter the growth rate is $1+i_1 = \frac{80,800}{75,000} = 1.07733$. The new fund balance after the 4/1 deposit is $95,800$, and this shrinks to $93,700$ on 6/30. Thus the second quarter growth rate is $1+i_2 = \frac{93,700}{95,800} = .97807$. The new fund after the 7/1 benefit payment is $81,500$, and this grows to $(73,200+10,100) = 83,300$ just before the 9/29 benefit payment. Thus the third quarter growth rate is $1+i_3 = \frac{83,300}{81,500} = 1.02209$. The 9/30 fund of $73,200$ grows to $(77,600-2700) = 74,900$ just before the 12/31 deposit, so the fourth quarter growth rate is $1+i_4 = \frac{74,900}{73,200} = 1.02322$. Then the time-weighted rate of return for 1988 is $i = (1.07733)(.97807)(1.02209)(1.02322) - 1 = .10199$, ANSWER E.

S-40. The problem is illustrated on the following diagram: (amounts in thousands)



Let i_1, i_2, i_3 and i_4 be the rates of return for the four quarters of 1989.

The fund balance *just before* the 3/31 deposit is $110,000$, so the first quarter interest growth is from $100,000$ to $110,000$, so that $(1+i_1) = \frac{110,000}{100,000} = \frac{110}{100}$.
The fund balance *just before* the 6/30 deposit and withdrawal is $(95,000+40,000-10,000) = 125,000$, so the second quarter interest growth is from $130,000$ to $125,000$, so that $(1+i_2) = \frac{125,000}{130,000} = \frac{125}{130}$.
The fund balance *just before* the 9/30 deposit is $110,000$, so the third quarter interest growth is from $95,000$ to $110,000$, so that $(1+i_3) = \frac{110}{95}$.
The fund balance *just before* the 12/31 deposit and withdrawal is $(150,000+5000-10,000) = 145,000$, so the fourth quarter interest growth is from $130,000$ to $145,000$, so that $(1+i_4) = \frac{145}{130}$. Therefore
$i = (1+i_1)(1+i_2)(1+i_3)(1+i_4) - 1 = \left(\frac{110}{100}\right)\left(\frac{125}{130}\right)\left(\frac{110}{95}\right)\left(\frac{145}{130}\right) - 1 = .3660$, ANSWER D.

S-41. I. From 1/1/90 to 3/31/90 the fund grows from $100,000$ to $110,000$ (before the contribution), for a growth factor of $1+i_1 = \frac{110,000}{100,000} = 1.10$. From 4/1/90 to 9/30/90 the fund grows from $160,000$ to $160,000$ (before the payment), for a growth factor of $1+i_2 = 1.00$.
From 10/1/90 to 12/31/90 the fund grows from $130,000$ to $130,000$ for a growth factor of $1+i_3 = 1.00$. Then clearly $1+i = 1+i_1$, so $\underline{i_I = .10}$.

II. The fund grew from $100,000$ to $130,000$ over the year, with a net contribution of $20,000$ (net over payments). The remaining $10,000$ of growth is therefore due to interest. Under this method the rate of return is given by the ratio of interest earned to average amount invested, assuming the interest to be added at year end.
Thus we have $i_{II} = \frac{I}{\frac{1}{2}(A+B-I)} = \frac{10,000}{\frac{1}{2}(100,000+130,000-10,000)} = \underline{.09091}$.

S-41. (cont.)

III. In this method we accumulate the initial fund and the contributions and payments to year-end, using simple interest. Thus we have

$100,000(1+i)+50,000(1+.75i) \;-\; 30,000(1+.25i) \;=\; 130,000,$ leading to

$$i_{III} \;=\; \frac{10,000}{100,000+.75(50,000)-.25(30,000)} \;=\; \underline{.07692}.$$

Thus we find I $>$ II $>$ III, ANSWER A.

S-42.I. Time-weighted:

Over the four quarters of 1991 the respective fund growths are (in thousands) from 200 to 210, from 200 to 216, from 286 to 286, and from 276 to 270.

Then $i \;=\; \left(\frac{210}{200}\right)\left(\frac{216}{200}\right)\left(\frac{286}{286}\right)\left(\frac{270}{276}\right) - 1 \;=\; .10934.$

II. Dollar-weighted:

$200{,}000(1+i) - 10{,}000(1+i)^{3/4} + 70{,}000(1+i)^{1/2} - 10{,}000(1+i)^{1/4} = 270{,}000$

Assuming simple interest we have

$200{,}000(1+i) - \left(1+\frac{3}{4}i\right) + 70{,}000\left(1+\frac{1}{2}\,i\right) - 10{,}000\left(1+\frac{1}{4}i\right) = 270{,}000,$

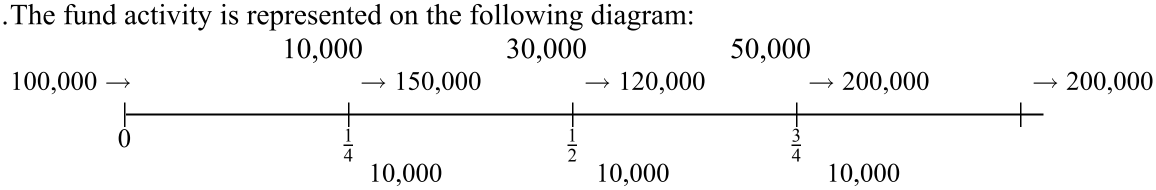
leading to $i = \dfrac{20{,}000}{200{,}000 - \frac{3}{4}(10{,}000) + \frac{1}{2}(70{,}000) - \frac{1}{4}(10{,}000)} \;=\; .08\dot{8}.$

III. There is a net contribution to the fund of 40,000. Assuming a uniform distribution of contributions and payments, we can consider the 40,000 net contribution to be made at mid-year. Then $200{,}000(1+i) + 40{,}000\left(1+\frac{1}{2}i\right) = 260{,}000$, so

$$i = \frac{20{,}000}{200{,}000 + \frac{1}{2}(40{,}000)} = .09091.$$

Therefore we find I $>$ III $>$ II, ANSWER A.

S-43. The fund activity is represented on the following diagram:



The dollar-weighted method uses an equation of value for the fund activity, but does not use the intermediate fund balances. Thus we have

$$100{,}000(1+i) + 20{,}000(1+i)^{1/2} + 40{,}000(1+i)^{1/4} \;=\; 200{,}000.$$

Approximating the compound interest functions by simple interest we have

$$100(1+i) + 20\left(1+\tfrac{1}{2}i\right) + 40\left(1+\tfrac{1}{4}i\right) \;=\; 200,$$

which solves for $i_D \;=\; \dfrac{200-160}{120} \;=\; .3333\dot{3}.$

For the time-weighted method, we need the fund balances *before* each fund activity as well as after. These pre-activity balances are easily found to be 150,000 at $t=\frac{1}{4}$, 100,000 at $t=\frac{1}{2}$, and 160,000 at $t=\frac{3}{4}$. Then we have

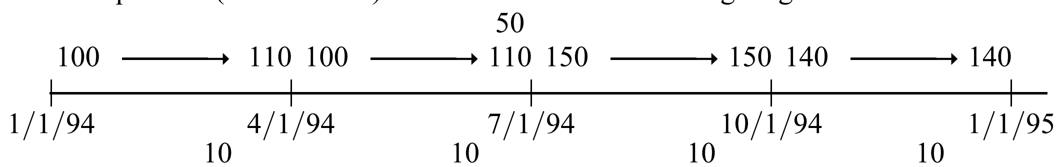
$$1+i \;=\; \left(\frac{150{,}000}{100{,}000}\right)\left(\frac{100{,}000}{150{,}000}\right)\left(\frac{160{,}000}{120{,}000}\right)\left(\frac{200{,}000}{200{,}000}\right),$$

which solves for $i_T = .3333\dot{3}$. Then the difference is $i_D - i_T \;=\; 0$, ANSWER A.

S-44.I. The fund balances *just before* each contribution/payment point are as follows:

Date	Balance
1/1/94	100,000
4/1/94	110,000
7/1/94	110,000
10/1/94	150,000
1/1/95	140,000

The balances *just after* the contribution/payment points are given in the question. All the data for this question (in thousands) is illustrated on the following diagram:



The time-weighted rate of return is given by

$$i_I = \left(\frac{110}{100}\right)\left(\frac{110}{100}\right)\left(\frac{150}{150}\right)\left(\frac{140}{140}\right) - 1 = .21000.$$

II. To find the dollar-weighted rate of return we ignore the intermediate fund balances, and write the equation of value

$$100(1+i) - 10(1+i)^{3/4} - 10(1+i)^{1/2} + 50(1+i)^{1/2} - 10(1+i)^{1/4} = 140.$$

Making the simple interest assumption and solving for i, we find

$$i_{II} = \frac{140 - 100 + 10 + 10 - 50 + 10}{100 - \frac{3}{4}(10) - \frac{1}{2}(10) + \frac{1}{2}(50) - \frac{1}{4}(10)} = .18182.$$

III. The cash flow in the first quarter, if uniformly distributed, can be viewed as a withdrawal of 10 in the middle of the quarter. Thus the effective rate for the first quarter is found from

$$100(1+i_1) - 10(1+i_1)^{1/2} = 100,$$

leading to $i_1 = \dfrac{100 - 100 + 10}{100 - \frac{1}{2}(10)} = .10526.$

In the second quarter we view a net deposit of 40 in the middle of the quarter. The effective rate for the second quarter is found from

$$100(1+i_2) + 40(1+i_2)^{1/2} = 150,$$

leading to $i_2 = \dfrac{150 - 100 - 40}{100 + \frac{1}{2}(40)} = .08333.$

In similar manner we find $i_3 = \dfrac{140 - 150 + 10}{150 - \frac{1}{2}(10)} = .00000,$

and $i_4 = \dfrac{130 - 140 + 10}{140 - \frac{1}{2}(10)} = .00000.$ Then the annual effective rate is

$$i_{III} = (1.10526)(1.08333)(1.00000)(1.00000) - 1 = .19736.$$

Finally we find $i_I > i_{III} > i_{II}$, ANSWER B.

S-45. For the dollar-weighted rate of return, an equation of value is used to move from the 1/1 fund balance to the 12/31 fund balance. Thus we have

$$1{,}200{,}000(1+i) + (100{,}000-150{,}000)(1+i)^{3/4} + (250{,}000-150{,}000)(1+i)^{5/12} + (350{,}000-150{,}000)(1+i)^{2/12} = 1{,}800{,}000.$$

Using simple interest to solve the equation for i we find

$$1{,}200{,}000(1+i) - 50{,}000\left(1+\frac{3}{4}i\right) + 100{,}000\left(1+\frac{5}{12}i\right) + 200{,}000\left(1+\frac{2}{12}i\right) = 1{,}800{,}000,$$

which solves for

$$i_D = \frac{1{,}800{,}000 - 1{,}200{,}000 + 50{,}000 - 100{,}000 - 200{,}000}{1{,}200{,}000 - \frac{3}{4}(50{,}000) + \frac{5}{12}(100{,}000) + \frac{2}{12}(200{,}000)} = .28282.$$

For the time-weighted rate of return (see Parmenter Section 2.3), the year is partitioned at each cash flow point and the effective rate of growth is obtained for each subinterval of the year. From 1/1 to 3/31 the fund grows from 1,200,000 to 1,450,000 (just before the net 50,000 withdrawal). From 4/1 to 7/31 the fund grows from 1,400,000 to 900,000 (just before the net 100,000 deposit). From 8/1 to 10/31 the fund grows from 1,000,000 to 1,400,000 (just before the net 200,000 deposit). From 11/1 to 12/31 the fund grows from 1,600,000 to 1,800,000. Then the time-weighted rate of return is

$$i_T = \left(\frac{1450}{1200}\right)\left(\frac{900}{1400}\right)\left(\frac{1400}{1000}\right)\left(\frac{1800}{1600}\right) - 1 = .22344.$$

Finally the difference is

$$.28282 - .22344 = .05938, \text{ ANSWER D.}$$

S-46. The effective quarterly interest rate for the annuity is $j = \frac{.05}{4} = .0125$, so the purchase price of the annuity is

$$P = 1000\ddot{a}_{\overline{80}|.0125} = 51{,}016.44.$$

The quarterly annuity payments are then reinvested at quarterly effective rate $j = (1.05)^{1/2} - 1 = .02470$, accumulating to

$$S = 1000\ddot{s}_{\overline{80}|.02470} = 250{,}624.02.$$

Thus we see that the original investment of 51,016.44 has grown to 250,624.02 over 20 years, for an effective annual compound rate of

$$i = \left(\frac{250{,}624.02}{51{,}016.44}\right)^{1/20} - 1$$

$$= .08284, \text{ ANSWER C.}$$

S-47. The dollar-weighted rate of return is found from the equation

$$1{,}000{,}000(1+i) - 50{,}000(1+i)^{1/2} + 100{,}000(1+i)^r = 1{,}150{,}000.$$

Making the usual simple interest approximation we have

$$i\big[1{,}000{,}000 - 50{,}000(\tfrac{1}{2}) + 100{,}000(r)\big] = 100{,}000$$

so that

$$i_D = \frac{100{,}000}{975{,}000 + 100{,}000r} = \frac{4}{39 + 4r}.$$

The time-weighted rate does not depend on the timing of the cash flows, so we have

$$i_T = \left(\frac{1{,}030{,}000}{1{,}000{,}000}\right)\left(\frac{1{,}025{,}000}{980{,}000}\right)\left(\frac{1{,}150{,}000}{1{,}125{,}000}\right) - 1 = .101236.$$

Equating $i_D = i_T$ we easily find $r = .12790$, which is the fraction of the year remaining after date X. Thus we have $r = .1279(365) = 46.7$ days, which is approximately November 15, ANSWER E.

S-48. The dollar-weighted rate of return is the value of i that satisfies the equation

$$60{,}000(1+i) + 30{,}000\left(1+\tfrac{2}{12}i\right) - 1{,}000(1+i) - 1000\left(1+\tfrac{11}{12}i\right) - \cdots - 1000\left(1+\tfrac{1}{12}i\right) = B,$$

where B is the actual fund value on 12/31/97. We know that $i = .07$, so we can solve this equation for B, obtaining

$$B = 60{,}000 + 30{,}000 - 12{,}000 + .07\left[60{,}000 + \tfrac{2}{12}(30{,}000) - 1000\left(1+\tfrac{11}{12}+\cdots+\tfrac{1}{12}\right)\right]$$

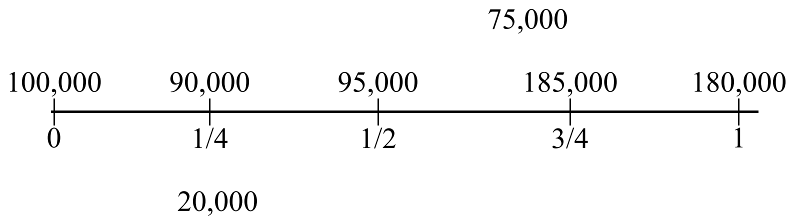
$$= 78{,}000 + .07\left[60{,}000+\tfrac{2}{12}(30{,}000)-\tfrac{78}{12}(1{,}000)\right] = 82{,}095.00.$$

Similarly, the expected fund value on 12/31/97, denoted B', is calculated the same way using 6% in place of 7%. Thus we have

$$B' = 78{,}000 + .06\left[60{,}000+\tfrac{2}{12}(30{,}000)-\tfrac{78}{12}(1{,}000)\right] = 81{,}510.00.$$

Then the investment gain is $B - B' = 585$, ANSWER A.

S-49. The fund activity and fund values are shown on the following diagram:



$$t = \left(\frac{110}{100}\right)\left(\frac{95}{90}\right)\left(\frac{110}{95}\right)\left(\frac{180}{185}\right) - 1 = .30811$$

In general, we can write

$$100(1+i) - 20(1+i)^{3/4} + 75(1+i)^{1/4} = 180.$$

Using "simple interest" we have

$$i[100 - \tfrac{3}{4}(20) + \tfrac{1}{4}(75)] = 180 - 100 + 20 - 75$$

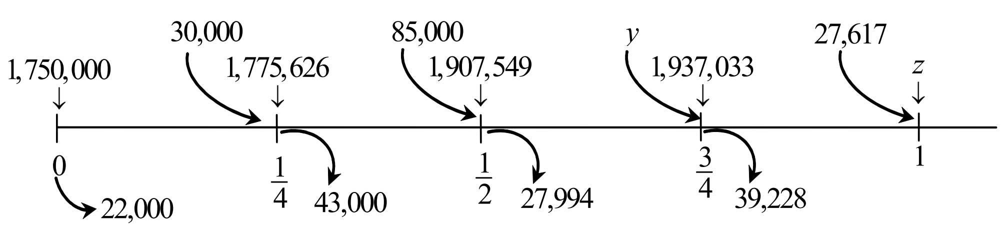
so

$$d = i = \frac{25}{103.75} = .24096.$$

If all cash flow is at mid-year, then we have $m = \dfrac{25}{100 + \frac{1}{2}(55)} = .19607.$

Thus we have $m < d < t$, ANSWER B.

S-50. On the following diagram, fund balances are located at the vertical arrows, whereas payments (out) and contributions (in) are located at the curved arrows.



The dollar-weighted rate of return is the solution of the equation.

$$1{,}750{,}000(1+i) - 22{,}000(1+i) + (30{,}000-43{,}000)\left(1+\tfrac{3}{4}i\right)$$
$$+ (85{,}000-27{,}994)\left(1+\tfrac{1}{2}i\right) + (y-39{,}228)\left(1+\tfrac{1}{4}i\right) + 27{,}617 = z,$$

which simplifies to $1{,}760{,}395 + y + i[1{,}736{,}946 + .25y] = z$, or $i = \dfrac{z - 1{,}760{,}395 - y}{1{,}736{,}946 + .25y}$.

S-50. (cont.)

The value of y is found by looking at Quarter 3 only:

$$\frac{1{,}937{,}033 - y}{1{,}907{,}549 - 27{,}994} = 1 - .0226 = .9774$$

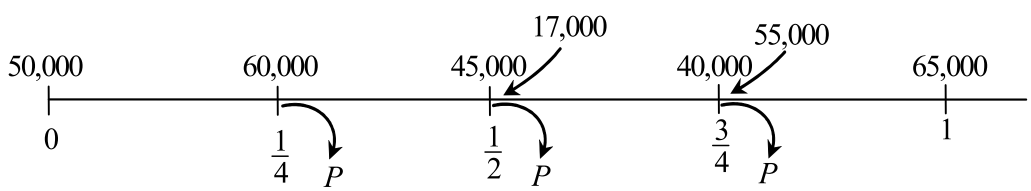
This equation solves for $y = 99{,}956$. The value of z is found from the calculation of the time-weighted rate of return:

$$(1.0102)(1.0519)(.9774)\left(\frac{z - 27{,}617}{1{,}937{,}033 - 39{,}228}\right) = 1.0879$$

This equation solves for $z = 2{,}015{,}480$. Finally,

$$i = \frac{2{,}015{,}480 - 1{,}760{,}395 - 99{,}956}{1{,}736{,}946 + .25(99{,}956)} = .08804, \text{ ANSWER D.}$$

S-51. The data are represented on the following diagram:



Using the dollar-weighted method, which ignores the intermediate fund balances, we have

$$50{,}000(1+i) - P(1+i)^{3/4} + (17{,}000-P)(1+i)^{1/2} + (55{,}000-P)(1+i)^{1/4} = 65{,}000.$$

Using a linear approximation for the exponential functions, the value of i turns out to be .07. Thus we have

$$50{,}000(1.07) + 17{,}000\left[1+\tfrac{1}{2}(.07)\right] + 55{,}000\left[1+\tfrac{1}{4}(.07)\right] - 65{,}000$$
$$= P\left\{\left[1+\tfrac{3}{4}(.07)\right] + \left[1+\tfrac{1}{2}(.07)\right] + \left[1+\tfrac{1}{4}(.07)\right]\right\},$$

which solves for $P = \dfrac{62{,}057.50}{3.105} = 19{,}986.31$.

Then the time-weighted rate of return is

$$i_T = \left(\frac{60}{50}\right)\left(\frac{45}{60-19.98631}\right)\left(\frac{40}{45+17-19.98631}\right)\left(\frac{65}{40+55-19.98631}\right) - 1 = .11333,$$

ANSWER D.