FINANCIAL MATHEMATICS

Theory and Practice

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FINANCIAL MATHEMATICS:
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Financial Mathematics: Theory and Practice

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Preface

Welcome to an alternative approach to the teaching and learning of financial mathematics!

We have a combined 68 years of teaching experience in actuarial science in general, and financial mathematics in particular. (Robert is now retired from the University of Waterloo, where he taught actuarial science for 40 years, and Steve is currently in his 28th year of teaching actuarial science and statistics at the University of Western Ontario.) In those 68 years, we have always lamented the fact that while there were multiple references available to our students, we had to spend extra time in preparation of our classes and assignments to fill in certain pedagogical deficiencies in any one of these references. Thus, our desire to enter the textbook listings for this discipline.

Our textbook is designed for mathematics / actuarial students with a strong grounding in, and understanding of, a basic first-year calculus curriculum. Our target audience contains students who plan to write the established actuarial examinations as made available around the world and, some day, to have a career in the actuarial or finance professions.

This textbook is designed to provide the reader with a generic approach to understanding financial mathematics with respect to a wide range of financial transactions, including annuities, mortgages, personal loans, and bonds, as well as a limited knowledge of advanced topics such as duration and immunization. All examples are solved from absolute first principles and we propose that any student should solve all financial mathematics problems in the same way; that is, from absolute first principles.

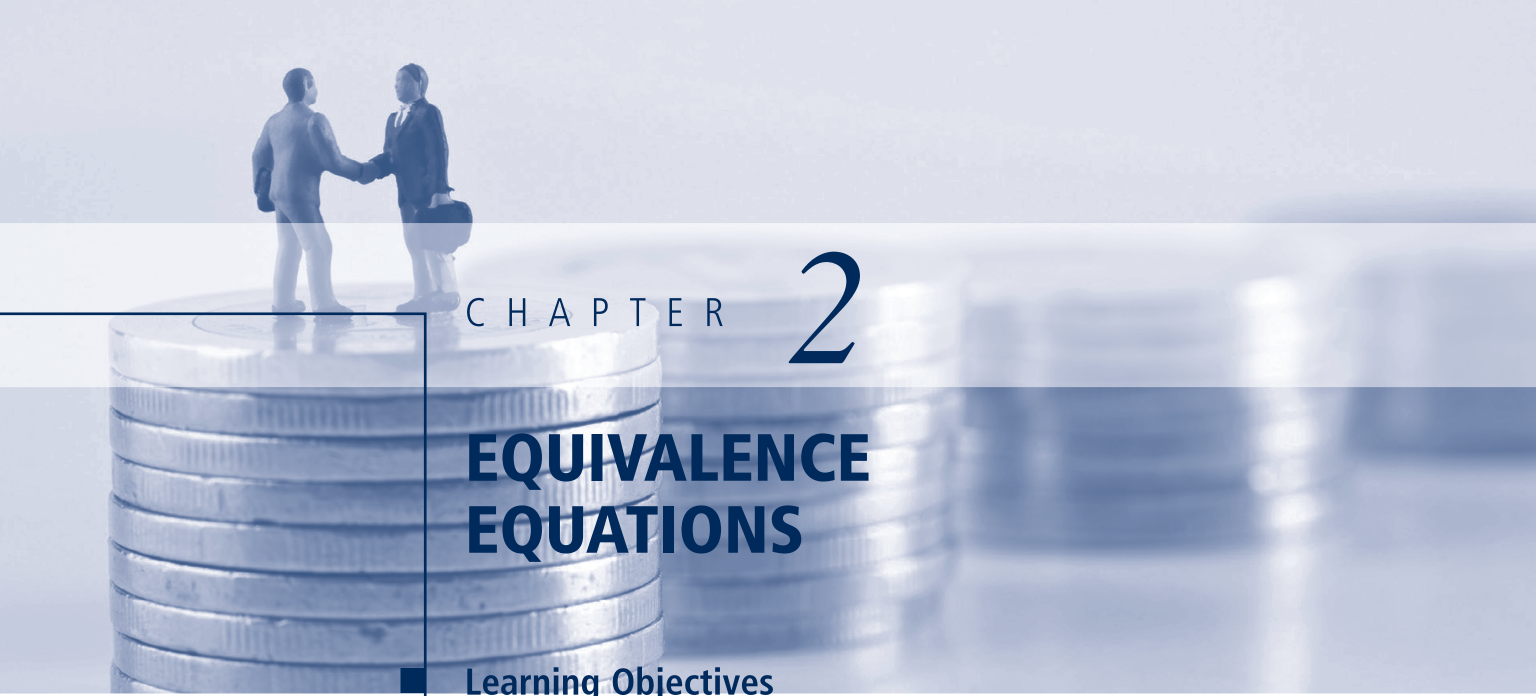
As previously stated, this textbook assumes a knowledge of introductory calculus, so problems are posed whose solutions may require the use of derivatives, or integrals, or both. A comfort with exponents and logarithms is also assumed.

We would be remiss if we did not thank Jeff Snook, Kimberly Veevers, and Stephanie Gay at McGraw-Hill Ryerson, and Javid Ali at the University of Waterloo, for their essential assistance in creating this textbook.

This textbook acknowledges the existence of modern financial calculators and spreadsheets and uses these technologies where appropriate. However, in each and every instance, our goal is to have the student be equipped to understand each solution from fundamental principles.

Our passion for, and interest in, financial mathematics remains undiminished. It is our sincere hope that this textbook provides the student with an improved understanding of the fascinating calculations that underlie most financial transactions.

Robert Brown, Victoria, British Columbia
Steve Kopp, London, Ontario



CHAPTER 2

EQUIVALENCE EQUATIONS

Learning Objectives

In Chapter 1, we learned how to perform financial calculations using simple and compound interest and using simple and compound discount. We also worked with nominal rates of interest and discount. Most financial transactions involve compound interest rates. Beginning with this chapter, the rest of this textbook is devoted to financial calculations using compound interest, unless stated otherwise.

Upon completing this chapter, you will be able to do the following:

- Calculate equivalent debts and payments using an equation of value.
- Determine how long it will take to earn a certain amount of interest.
- Calculate the rate of return on an investment.
- Solve non-financial problems that involve geometric growth.
- Calculate the price of Treasury bills in both Canada and the United States.

Section 2.1

Equations of Value

All financial decisions must take into account the basic idea that **money has time value**. In other words, receiving \$100 today is not the same as receiving \$100 one year ago, nor receiving \$100 one year from now if there is a positive interest rate. In a financial transaction involving money due on different dates, every sum of money should have an attached date, the date on which it falls due. That is, the mathematics of finance deals with **dated values**. This is one of the most important facts in the mathematics of finance.

Illustration: At a simple interest rate of 8%, \$100 due in one year is considered to be equivalent to \$108 in two years since \$100 would accumulate to \$108 in one year. In the same way

$$\$100(1 + 0.08)^{-1} = \$92.59$$

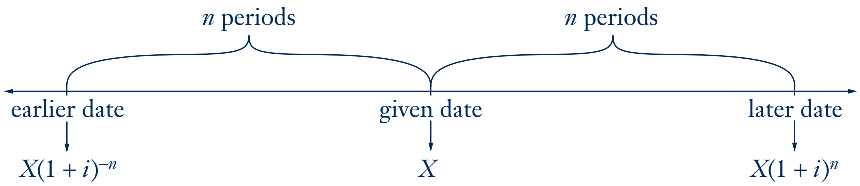
would be considered an equivalent sum today.

Another way to look at this is to suppose you were offered \$92.59 today, or \$100 one year from now, or \$108 two years from now. Which one would you choose? Most people would probably take the \$92.59 today because it is money in their hands. However, from a financial point of view, all three values are the same, or are equivalent, since you could take the \$92.59 and invest it for one year at 8%, after which you would have \$100. This \$100 could then be invested for one more year at 8% after which it would have accumulated to \$108. Note that the three dated values are not equivalent at some other rate of interest.

In general, we compare dated values by the following **definition of equivalence**:

> \$$X$ due on a given date is equivalent at a given compound interest rate i to Y due n periods later, if $Y = X(1 + i)^n$ or $X = Y(1 + i)^{-n}$

The following diagram illustrates dated values equivalent to a given dated value X.



Based on the time diagram above we can state the following simple rules:

> **1.** When we move money forward in time, we accumulate, i.e., we multiply the sum by an accumulation factor $(1 + i)^n$.
>
> **2.** When we move backward in time, we discount, i.e., we multiply the sum by a discount factor $(1 + i)^{-n}$.

The following property of equivalent dated values, called the *property of transitivity*, holds under compound interest:

At a given compound interest rate, if X is equivalent to Y, and Y is equivalent to Z, then X is equivalent to Z.

To prove this property we arrange our data on a time diagram.

If X is equivalent to Y, then $Y = X(1 + i)^{n_2 - n_1}$
If Y is equivalent to Z, then $Z = Y(1 + i)^{n_3 - n_2}$

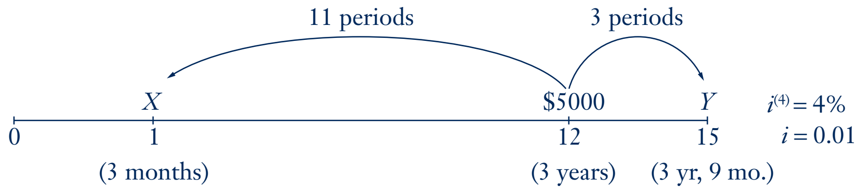
Eliminating Y from the second equation we obtain

$$Z = X(1 + i)^{n_2 - n_1}(1 + i)^{n_3 - n_2} = X(1 + i)^{n_3 - n_1}$$

Thus Z is equivalent to X.

EXAMPLE 1 A debt of \$5000 is due at the end of 3 years. Determine an equivalent debt due at the end of a) 3 months; b) 3 years 9 months, if $i^{(4)} = 4\%$.

Solution We arrange the data on a time diagram below.



By definition of equivalence

$$X = \$5000(1.01)^{-11} = \$4481.62$$
$$Y = \$5000(1.01)^{3} = \$5151.51$$

Note that X and Y are equivalent by verifying

$$Y = X(1.01)^{14} \quad \text{or} \quad \$5151.51 = \$4481.62(1.01)^{14}$$

The sum of a set of dated values, due on different dates, has no meaning. We have to replace all the dated values by equivalent dated values, due on the same date. The sum of the equivalent values is called the **dated value of the set**.

At compound interest the following property is true: *The various dated values of the same set are equivalent.* The proof is left as an exercise. See problem 1 of Part B of Exercise 2.1.

EXAMPLE 2 A person owes \$200 due in 6 months and \$300 due in 15 months. What single payment a) now, b) in 12 months, will liquidate these obligations if money is worth $i^{(12)} = 6\%$?

Solution We arrange the data on the diagram below. Let X be the single payment due now and Y be the single payment due in 12 months.

To calculate the equivalent dated value X, we must discount the \$200 debt for 6 months and discount the \$300 debt for 15 months.

$$X = \$200(1.005)^{-6} + \$300(1.005)^{-15} = \$194.10 + \$278.38 = \$472.48$$

To calculate the equivalent dated value Y, we must accumulate the \$200 debt from time 6 to time 12, or 6 periods, and discount the \$300 debt from time 15 to time 12, or 3 periods.

$$Y = \$200(1.005)^{6} + \$300(1.005)^{-3} = \$206.08 + \$295.54 = \$501.62$$

We can verify the property of transitivity of X and Y by showing that

$$Y = X(1.005)^{12} = \$472.48(1.005)^{12} = \$501.62$$
$$\text{or} \qquad X = Y(1.005)^{-12} = \$501.62(1.005)^{-12} = \$472.48$$

One of the most important problems in the mathematics of finance is the replacing of a given set of payments by an equivalent set. Two sets of payments are equivalent at a given compound interest rate if the dated values of the sets, on any common date, are equal. An equation stating that the dated values, on a common date, of two sets of payments are equal is called an **equation of value** or an **equation of equivalence**. The date used is called the **focal date** or the **comparison date** or the **valuation date**.

The procedure for setting up and solving an equation of value is stated below.

Step 1 Draw a good time diagram showing the dated values of debts on one side of the time line and the dated values of payments on the other side. The times on the line should be stated in terms of interest compounding periods.

Step 2 Select one, and only one, focal date and accumulate/discount all dated values to the focal date using the specified compound interest rate.

Step 3 Set up an equation of value at the focal date:

dated value of payments = dated value of debts.

Step 4 Solve the equation of value using methods of algebra.

The following examples illustrate the use of equations of value in the mathematics of finance.

EXAMPLE 3 A debt of \$1000 with interest at $i^{(4)} = 10\%$ will be repaid by a payment of \$200 at the end of 3 months and three equal payments at the ends of 6, 9, and 12 months. What will these payments be?

Solution We arrange all the dated values on a time diagram.

debt:	\$1000					$i^{(4)} = 10\%$
	0	1	2	3	4	$i = 0.025$
payments:		\$200	X	X	X	

Any date can be selected as a focal date. We show the calculation using the end of 12 months and the present time.

Equation of value at the end of 12 months:

$$\text{dated value of the payments} = \text{dated value of the debts}$$
$$200(1.025)^3 + X(1.025)^2 + X(1.025)^1 + X = 1000(1.025)^4$$
$$215.38 + 1.050625X + 1.025X + X = 1103.81$$
$$3.075625X = 888.43$$
$$X \doteq 288.86$$

Equation of value at the present time:

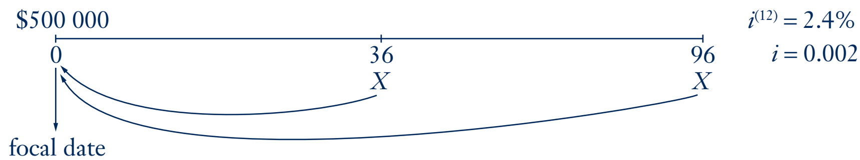
$$200(1.025)^{-1} + X(1.025)^{-2} + X(1.025)^{-3} + X(1.025)^{-4} = 1000.00$$
$$195.12 + 0.951814396X + 0.928599411X + 0.905950645X = 1000.00$$
$$2.786364452X = 804.88$$
$$X \doteq 288.86$$

CALCULATION TIP:

Choose a convenient focal date, one that simplifies your calculations, when using equations of value at compound interest.

EXAMPLE 4 A man leaves an estate of \$500 000 that is invested at $i^{(12)} = 2.4\%$. At the time of his death, he has two children aged 13 and 18. Each child is to receive an equal amount from the estate when they reach age 21. How much does each child get?

Solution The older child will get \$$X$ in 3 years (36 months); the younger child will get \$$X$ in 8 years (96 months). We arrange the dated values on a time diagram.



Equation of value at present:

$$X(1.002)^{-36} + X(1.002)^{-96} = 500\,000$$
$$0.930597807X + 0.825465131X = 500\,000$$
$$1.756062939X = 500\,000$$
$$X \doteq \$284\,727.84$$

Each child will receive \$284 727.84

The following calculation checks the correctness of the answer:

$$\text{Amount in fund at the end of 3 years} = \$500\,000(1.002)^{36} = \$537\,289.04$$
$$\text{Payment to the older child} = \$284\,727.84$$
$$\text{Balance in the fund} = \$252\,561.20$$
$$\text{Amount in fund after 5 more years} = \$252\,561.20(1.002)^{60} = \$284\,727.83$$

The 1-cent difference is due to rounding.

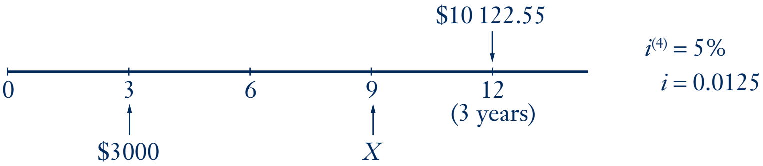
EXAMPLE 5 Stephanie owes \$8000 due at the end of 3 years with interest at $i^{(2)} = 8\%$. The lender agrees to allow her to pay back the loan early with a payment of \$3000 at the end of 9 months and \$$X$ at the end of 27 months. If the lender can reinvest any payments at $i^{(4)} = 5\%$, what is X?

Solution In this example, there are two interest rates: 8% is the interest rate on the original loan, while 5% is the rate that will be used in the equation of value.

First we need to calculate the maturity value of the loan, which is due in 3 years (6 half-years):

$$S = 8000(1.04)^6 = 10\ 122.55$$

We now set up our time diagram, with \$10 122.55 as the original debt and using quarter-years as the times.



Equation of value at time 9:

$$\begin{aligned} 3000(1.0125)^6 + X &= 10122.55(1.0125)^{-3} \\ 3232.149542 + X &= 9752.250203 \\ X &\doteq 6520.10 \end{aligned}$$

A payment of \$3000 at the end of 9 months and \$6520.10 at the end of 27 months is equivalent to a debt of \$10 122.55 at the end of 3 years.

To check this is the correct answer:

Payment at end of 9 month, invested at $i^{(4)} = 5\%$, $\$3000(1.0125)^6 =$	\$3232.15
Payment at end of 27 months =	\$6520.10
Balance	\$9752.25

Balance invested at $i^{(4)} = 5\%$ to end of 3 years = $\$9752.25(1.0125)^3 = \$10122.55$ which is equal to the original debt owed at the end of 3 years.

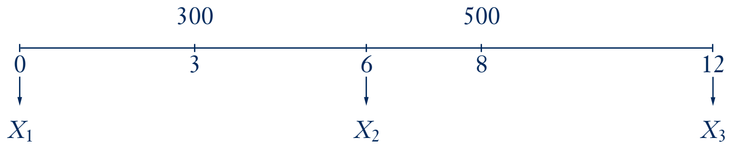
An equation of value can be set up for problems involving simple interest or simple discount. The procedure is exactly the same, except the choice of focal date will change your final answer. That is because the property of transitivity does not hold when using simple interest or simple discount accumulation functions or discount functions. Thus, both parties to a financial transaction must agree on which focal date to use.

OBSERVATION:

In mathematics, an equivalence relation must satisfy the property of transitivity. Thus, strictly speaking, the equivalence of dated values at a simple interest rate is not an equivalence relation, and that is why when using simple interest you will get a different answer depending on what focal date is used. However, the choice of focal date under compound interest will not affect the final answer because the equivalence of dated values at a compound interest rate is an equivalence relation.

EXAMPLE 6 A person owes \$300 due in 3 months and \$500 due in 8 months. The lender agrees to allow the person to pay off these two debts with a single payment. What single payment a) now; b) in 6 months; c) in 1 year will liquidate these obligations if money is worth 8% simple interest?

Solution

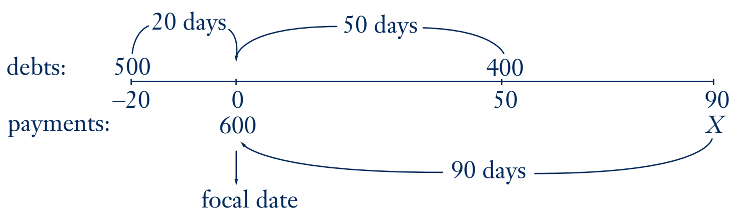


We calculate equivalent dated values of both obligations at the three different times and arrange them in the table below.

Obligations	Now	In 6 Months	In 1 Year
First	$300\left[1 + (0.08)\left(\frac{3}{12}\right)\right]^{-1} = 294.12$	$300\left[1 + (0.08)\left(\frac{3}{12}\right)\right] = 306.00$	$300\left[1 + (0.08)\left(\frac{9}{12}\right)\right] = 318.00$
Second	$500\left[1 + (0.08)\left(\frac{8}{12}\right)\right]^{-1} = 474.68$	$500\left[1 + (0.08)\left(\frac{2}{12}\right)\right]^{-1} = 493.42$	$500\left[1 + (0.08)\left(\frac{4}{12}\right)\right] = 513.33$
Sum	$X_1 = 768.80$	$X_2 = 799.42$	$X_3 = 831.33$

EXAMPLE 7 Debts of \$500 due 20 days ago and \$400 due in 50 days are to be settled by a payment of \$600 now and a final payment 90 days from now. Determine the value of the final payment at a simple interest rate of 11% with a focal date at the present.

Solution We arrange the dated values on a time diagram.



Equation of value at the present time:

dated value of payments = dated value of debts

$$X[1 + (0.11)(\tfrac{90}{365})]^{-1} + 600 = 500[1 + (0.11)(\tfrac{20}{365})] + 400[1 + (0.11)(\tfrac{50}{365})]^{-1}$$
$$0.973592958X + 600 = 503.01 + 394.06$$
$$0.973592958X = 297.07$$
$$X \doteq 305.13$$

The final payment to be made in 90 days is \$305.13.

Exercise 2.1

Part A

1. If money is worth $i^{(4)} = 6\%$ determine the sum of money due at the end of 15 years equivalent to \$1000 due at the end of 6 years.

2. What sum of money, due at the end of 5 years, is equivalent to \$1800 due at the end of 12 years if money is worth $i^{(2)} = 11\frac{3}{4}\%$?

3. An obligation of \$2500 falls due at the end of 7 years. Determine an equivalent debt at the end of a) 3 years and b) 10 years, if $i^{(12)} = 9\%$.

4. One thousand dollars is due at the end of 2 years and \$1500 at the end of 4 years. If money is worth $i^{(4)} = 8\%$, calculate an equivalent single amount at the end of 3 years.

5. Eight hundred dollars is due at the end of 4 years and \$700 at the end of 8 years. If money is worth $i^{(12)} = 3\%$, determine an equivalent single amount at a) the end of 2 years; b) the end of 6 years; c) the end of 10 years. Show your answers are equivalent.

6. A debt of \$2000 is due at the end of 8 years. If \$1000 is paid at the end of 3 years, what single payment at the end of 7 years would liquidate the debt if money is worth $i^{(2)} = 7\%$?

7. A person borrows \$4000 at $i^{(4)} = 6\%$. He promises to pay \$1000 at the end of one year, \$2000 at the end of 2 years, and the balance at the end of 3 years. What will the final payment be?

8. A consumer buys goods worth \$1500. She pays \$300 down and will pay \$500 at the end of 6 months. If the store charges $i^{(12)} = 18\%$ on the unpaid balance, what final payment will be necessary at the end of one year?

9. A debt of \$1000 is due at the end of 4 years. If money is worth $i^{(12)} = 8\%$, and \$375 is paid at the end of 1 year, what equal payments at the end of 2 and 3 years respectively would liquidate the debt?

10. On September 1, 2009, Paul borrowed \$3000, agreeing to pay interest at 12% compounded quarterly. He paid \$900 on March 1, 2010, and \$1200 on December 1, 2010.
a) What equal payments on June 1, 2011, and December 1, 2011, will be needed to settle the debt?
b) If Paul paid \$900 on March 1, 2010, \$1200 on December 1, 2010, and \$900 on March 1, 2011, what would be his outstanding balance on September 1, 2011?

11. A woman's bank account showed the following deposits and withdrawals:

	Deposits	Withdrawals
January 1, 2010	\$200	
July 1, 2010	\$150	
January 1, 2011		\$250
July 1, 2011	\$100	

If the account earns $i^{(2)} = 6\%$, determine the balance in the account on January 1, 2012.

12. Instead of paying \$400 at the end of 5 years and \$300 at the end of 10 years, a man agrees to pay \$$X$ at the end of 3 years and \$$2X$ at the end of 6 years. Determine X if $i^{(1)} = 10\%$.

13. A man stipulates in his will that $50 000 from his estate is to be placed in a fund from which his three children are to each receive an equal amount when they reach age 21. When the man dies, the children are ages 19, 15, and 13. If this fund earns interest at $i^{(2)} = 6\%$, how much does each receive?

14. To pay off a loan of $4000 at $i^{(12)} = 7.2\%$, Ms. Fil agrees to make three payments in 3, 7, and 12 months respectively. The second and third payment are to be double the first. What is the size of the first payment?

15. Mrs. Singh borrows $3000, due with interest at $i^{(12)} = 9\%$ in 2 years. The lender agrees to let Mrs. Singh repay the loan with a payment of $1000 in 6 months, $1500 in 12 months, and $$X$ in 30 months. If money is worth $i^{(2)} = 6\%$, what is the value of X?

16. Therèse owes $500 due in 4 months and $700 due in 9 months. If money is worth 7% simple interest, what single payment a) now; b) in 6 months; c) in 1 year, will liquidate these obligations?

17. Andrew owes Nicola $500 in 3 months and $200 in 6 months both due with simple interest at 6%. If Nicola accepts $300 now, how much will Andrew be required to repay at the end of 1 year, provided they agree to use a simple interest rate of 8% and a focal date at the end of 1 year?

18. A person borrows $1000 to be repaid with two equal instalments, one in 6 months, the other at the end of 1 year. What will be the size of these payments if the simple interest rate is 8% and the focal date is 1 year hence? What if the focal date is today?

19. Mrs. Adams has two options available in repaying a loan. She can pay $200 at the end of 5 months and $300 at the end of 10 months, or she can pay $$X$ at the end of 3 months and $$2X$ at the end of 6 months. Determine X if simple interest is at 12% and the focal date is 6 months hence and the options are equivalent. What is the answer if the focal date is 3 months hence and the options are equivalent?

Part B

1. a) Prove the following property in the case of a set of two dated values: the various dated values of the same set are equivalent at compound interest.
b) Show, algebraically, why this is not true for simple interest.

2. If money is worth $i^{(1)} = 8\%$, what single sum of money payable at the end of 2 years will equitably replace $1000 due today plus a $2000 debt due at the end of 4 years with interest at $12\frac{1}{2}\%$ per annum compounded semi-annually?

3. On January 1, 2010, Mr. Planz borrowed $5000 to be repaid in a lump-sum payment with interest at $i^{(4)} = 9\%$ on January 1, 2016. It is now January 1, 2012. Mr. Planz would like to pay $500 today and complete the liquidation with equal payments on January 1, 2014, and January 1, 2016. If money is now worth $i^{(4)} = 8\%$, what will these payments be?

4. You are given two loans, with each loan to be repaid by a single payment in the future. Each payment will include both principal and interest. The first loan is repaid by a $3000 payment at the end of 4 years. The interest is accrued at $i^{(2)} = 10\%$. The second loan is repaid by a $4000 payment at the end of 5 years. The interest is accrued at $i^{(2)} = 8\%$. These two loans are to be consolidated. The consolidated loan is to be repaid by two equal instalments of $$X$, with interest at $i^{(2)} = 12\%$. The first payment is due immediately and the second payment is due one year from now. Calculate X.

5. You are given the following data on three series of payments:

	Payment at End of Year			Accumulated Value at End of Year 18
	6	12	18	
Series A	240	200	300	X
Series B	0	360	700	$X + 100$
Series C	Y	600	0	X

Assume interest is compounded annually. Calculate Y.

Section 2.2 Determining the Rate and Time

Determining the Rate: When S, P, and n are given, we can substitute the given values into the fundamental compound interest formula $S = P(1 + i)^n$ and solve it for the unknown interest rate i.

$$S = P(1 + i)^n$$
$$(1 + i)^n = \frac{S}{P}$$
$$1 + i = \left(\frac{S}{P}\right)^{1/n}$$

$$\boxed{i = \left(\frac{S}{P}\right)^{1/n} - 1} \qquad \textbf{(8a)}$$

CALCULATION TIP:

Using the power key of our calculator, we calculate the exact value of i. The nominal annual rate of interest is determined by multiplying the effective rate i by the number of compounding periods per year, $i^{(m)} = mi$, and then rounding off to the nearest hundredth of a percent.

EXAMPLE 1 At what nominal rate $i^{(12)}$ will money triple itself in 12 years?

Solution We can use any sum of money as the principal. Let $P = x$, then $S = 3x$, and $n = 12 \times 12 = 144$.

Substituting in $S = P(1 + i)^n$ we obtain an equation for the unknown interest rate i per month

$$3x = x(1 + i)^{144}$$
$$(1 + i)^{144} = 3$$

Solving the exponential equation $(1 + i)^{144} = 3$ directly for i and then using a pocket calculator we have

$$(1 + i)^{144} = 3$$
$$1 + i = 3^{1/144}$$
$$i = 3^{1/144} - 1$$
$$i = 0.007658429$$
$$i^{(12)} = 12i = 0.091901147$$
$$i^{(12)} \doteq 9.19\%$$

Using a Financial Calculator

Many of the calculations that have been presented so far in this textbook can be performed using a financial calculator. However, for most of the exercises, it is easiest to set up an equation of value and solve for the answer using the symbols and formulas presented in this text, using a calculator to do only the basic math. There has been no real need for a financial calculator.

However, there are situations where a financial calculator can lead to quicker answers, and calculating the value of i is one such situation.

There are many different financial calculators on the market. Four good ones are: Sharp EL-733A, Hewlett Packard 10B, Texas Instruments BA-35, and Texas Instruments BA-II Plus. We will illustrate the calculations using the notation from the Texas Instruments BA-II Plus calculator.

In Example 1, we have $PV = -1$ (note that you need a negative sign in front of the present value), $FV = S = 3$, and $n = 144$ (the term is always entered as the number of interest periods). The steps are as follows:

−1 [PV] 3 [FV] 144 [N] [CPT] [I/Y]

0.765842888 (per month)

Note that the answer is given as a percentage. That is, the monthly rate is 0.765842888%. To obtain the nominal rate, we need to multiply by 12:

$$i^{(12)} = 12(0.76584288)\% = 9.190114656\% \doteq 9.19\%$$

EXAMPLE 2 Suppose you would like to earn \$600 of interest over 21 months on an initial investment of \$2500. What nominal rate of interest, $i^{(4)}$, is needed?

Solution We have $P = 2500$, $S = 2500 + 600 = 3100$, $n = 1.75 \times 4 = 7$ quarters, and $i = i^{(4)}/4$. Setting up the equation and solving, we get

$$\begin{aligned} 3100 &= 2500\,(1+i)^7 \\ 1+i &= (1.24)^{1/7} \\ i &= (1.24)^{1/7} - 1 \\ i &= 0.031207244 \\ i^{(4)} &= 4i = 0.124828974 \\ i^{(4)} &\doteq 12.48 \end{aligned}$$

Using the BA-II Plus calculator:

−2500 [PV] 3100 [FV] 7 [N] [CPT] [I/Y]

3.120724362 (per quarter)

To obtain the nominal rate,

$$i^{(4)} = 4(3.120724362)\% = 12.482897\% \doteq 12.48\%$$

Determining the Time: When S, P, and i are given, we can substitute the given values into the fundamental compound amount formula $S = P(1 + i)^n$ and solve it for the unknown n using logarithms.

In this textbook we assume that students have pocket calculators with a built-in common logarithmic function $\log x$ and its inverse function 10^x.

Solving the formula $S = P(1 + i)^n$ for n we obtain:

$$(1 + i)^n = \frac{S}{P}$$

$$n \log(1 + i) = \log\left(\frac{S}{P}\right)$$

$$\boxed{n = \frac{\log\left(\frac{S}{P}\right)}{\log(1 + i)}} \tag{8b}$$

EXAMPLE 3 How long will it take \$500 to accumulate to \$850 at $i^{(12)} = 12\%$?

Solution Let n represent the number of months, then we have

$$500(1.01)^n = 850$$
$$(1.01)^n = \tfrac{850}{500}$$
$$(1.01)^n = 1.7$$

We have

$$n \log 1.01 = \log 1.7$$
$$n = \frac{\log 1.7}{\log 1.01}$$
$$n = 53.3277 \text{ months}$$
$$n = 4.4440 \text{ years}$$
$$= 4 \text{ years}, \underbrace{\text{5 months},}_{\substack{0.4440 \times 12 \\ = 5.328}} \quad \underbrace{\text{10 days}}_{\substack{0.328 \times 30 \\ = 9.84 \doteq 10}}$$

Alternative Solution The accumulated value of \$500 for 53 periods at $i^{(12)} = 12\%$ is $500(1.01)^{53} =$ \$847.23. Now we calculate how long it will take \$847.23 to accumulate \$2.77 simple interest at rate 12%.

$$t = \frac{i}{Pr} = \frac{2.77}{847.23 \times 0.12} = 0.027245652 \text{ years} \doteq 10 \text{ days}$$

Thus the time is 4 years, 5 months, and 10 days.

Using a Financial Calculator

To calculate n using the BA-II Plus calculator, we have $PV = -500$, $FV = 850$, and $I/Y = 1$ (the interest rate is the effective rate per compounding period, entered as a number, not a decimal). The steps are as follows:

−500 [PV] 850 [FV] 1 [I/Y] [CPT] [N]

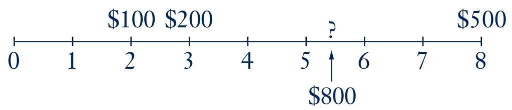
53.32769924 (months)

CALCULATION TIP:

When calculating unknown n from the fundamental compound interest formula **(7)**, use logarithms (that is, the exact method) in all exercises. The difference between the answers by the exact and the practical method is negligible.

EXAMPLE 4 Payments of \$100, \$200, and \$500 are paid at the end of years 2, 3, and 8 respectively. If $i^{(1)} = 5\%$ per annum effective, determine the time, t, at which a single payment of \$800 would be equivalent.

Solution



Using time 0 as the focal date.

$$\begin{aligned} 800(1.05)^{-t} &= 100(1.05)^{-2} + 200(1.05)^{-3} + 500(1.05)^{-8} \\ 800(1.05)^{-t} &= 601.89 \\ (1.05)^{t} &= \frac{800}{601.89} = 1.32915 \\ t &= \frac{\log\ 1.32915}{\log\ 1.05} = 5.832 \text{ years} \end{aligned}$$

EXAMPLE 5 Jerry invests \$1000 today in a fund earning $i^{(2)} = 5\%$. How long will it take for his money to double?

Solution We have $P = 1000$, $S = 2000$, $i = 0.025$ and will let n represent the number of half-years. Setting up the equation, we obtain

$$\begin{aligned} 1000(1.025)^{n} &= 2000 \\ (1.025)^{n} &= 2 \\ n \log\ 1.025 &= \log 2 \\ n &= \frac{\log 2}{\log 1.025} \\ n &= 28.0710 \text{ half years} \\ &= 14.0355 \text{ years} \\ &= 14 \text{ years, } \overbrace{\begin{matrix} 0.0355 \times 12 \\ = 0.426 \end{matrix}}^{0 \text{ months,}} \quad \overbrace{\begin{matrix} 0.4260 \times 30 \\ = 12.78 \doteq 13 \end{matrix}}^{13 \text{ days}} \end{aligned}$$

Note: If the interest rate had been $i^{(2)} = 10\%$, $n = 7$ years, 1 month, 8 days.
Using the BA-II Plus calculator and $i = 0.05/2 = 0.025$ per half-year,

$$-1000\ \boxed{PV}\ \ 2000\ \boxed{FV}\ \ 2.5\ \boxed{I/Y}\ \ \boxed{CPT}\ \ \boxed{N}$$

28.07103453 (half-years)

Using $i = 0.10/2 = 0.05$ per half-year,

$$-1000\ \boxed{PV}\ \ 2000\ \boxed{FV}\ \ 5\ \boxed{I/Y}\ \ \boxed{CPT}\ \ \boxed{N}$$

14.20669908 (half-year)

Rule of 70

A quick estimate for the amount of time needed for money to double in value can be obtained using the "Rule of 70." According to this rule, the number of interest periods needed for money to double in value is approximately equal to 70 divided by the effective rate of interest per period.

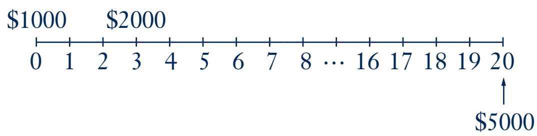
Using this rule in Example 5, we obtain,

i) If $i^{(2)} = 5\%$, then $i = 0.025$ and $n \doteq 70/2.5 = 28.0$ half-years. Compare this to the correct answer of 28.0710.

ii) If $i^{(2)} = 10\%$, then $i = 0.05$ and $n \doteq 70/5 = 14.0$ half-years. Compare this to the correct answer of 14.2067.

EXAMPLE 6 At what rate of interest, $i^{(2)}$, would an investment of \$1000 immediately and \$2000 in 3 years accumulate to \$5000 in 10 years?

Solution



Using the end of 10 years (time 20) as the focal date,

$$1000(1 + i)^{20} + 2000(1 + i)^{14} = 5000$$

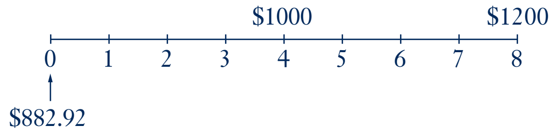
This cannot be solved algebraically. Instead you need to use "trial and error".

$i = \frac{i^{(2)}}{2}$	LHS
0.05	\$6613.16
0.04	\$5654.48
0.035	\$5227.18
0.033	\$5065.18
0.0322	\$5001.76
0.03218	\$5000.18

Rounding to five decimal points should be enough accuracy. Thus, the desired nominal rate of interest, $i^{(2)} = 2(0.03218) = 6.436\%$.

EXAMPLE 7 A deposit of \$882.92 today will allow you to withdraw \$1000 in 4 years and another \$1200 in 8 years. What effective rate of interest is assumed?

Solution



Using time 8 as our focal date:

$$882.92(1+i)^8 = 1000(1+i)^4 + 1200$$
$$882.92(1+i)^8 - 1000(1+i)^4 - 1200 = 0$$

This is a quadratic equation in the form $ax^2 + bx + c = 0$, where $x = (1+i)^4$, $a = 882.92$, $b = -1000$, and $c = -1200$.

Thus

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{1000 \pm \sqrt{(-1000)^2 - 4(882.92)(-1200)}}{2(882.92)} \\ &= -0.729778 \text{ or } +1.862383472 \end{aligned}$$

$$\begin{aligned} (1+i)^4 &= 1.862383472 \\ i &= (1.862383472)^{\frac{1}{4}} - 1 \\ &= 0.168200203 \\ &= 16.82\% \end{aligned}$$

Exercise 2.2

Part A

In problems 1 to 4 calculate the nominal rate of interest.

No.	Principal	Amount	Time	Conversion
1.	\$2000	\$3000.00	3 years 9 months	quarterly
2.	\$ 100	\$ 150.00	4 years 7 months	monthly
3.	\$ 200	\$ 600.00	15 years	annually
4.	\$1000	\$1581.72	3 years 6 months	semi-annually

In problems 5 to 8 determine the time.

No.	Principal	Amount	Interest Rate	Conversion
5.	\$2000	\$2800	4%	quarterly
6.	\$ 100	\$ 130	9%	semi-annually
7.	\$ 500	\$ 800	6%	monthly
8.	\$1800	\$2200	8%	quarterly

9. An investment fund advertises that it will guarantee to double your money in 10 years. What rate of interest $i^{(1)}$ is implied?

10. If an investment grows 50% in 4 years, what rate of interest $i^{(4)}$ is being earned?

11. From 2006 to 2011, the earnings per share of common stock of a company increased from \$4.71 to \$9.38. What was the compounded annual rate of increase?

12. At what rate $i^{(365)}$ will an investment of \$4000 grow to \$6000 in 3 years?

13. How long will it take to double your deposit in a savings account that accumulates at
a) $i^{(1)} = 4.56\%$? b) $i^{(365)} = 7\%$?
c) Redo a) and b) using the Rule of 70.

14. How long will it take for \$800 to grow to \$1500 in a fund earning interest at rate 9.8% compounded semi-annually?

15. How long will it take to increase your investment by 50% at rate 5% compounded daily?

16. The present value of \$1000 due in $2n$ years plus the present value of \$2000 due in $4n$ years is \$1388.68. If the interest rate is $i^{(12)} = 9.60\%$, what is the value of n?

17. You need to buy some furniture that costs \$12 900 in cash today. Alternatively, you can make a payment of \$4429 today, followed by \$4429 in 3 months, and \$4429 in 6 months. For the second option, the furniture company claims they are only charging you interest at a rate of $i^{(1)} = 3\%$ ($12\,900 \times 1.03 = 13\,287$, $\frac{13\,287}{3} =$ \$4429). What is the true rate of interest, $i^{(1)}$, you are being charged? [*Hint:* Calculate $i^{(4)}$ first.]

Part B

1. At a given rate of interest, $i^{(2)}$, money will double in value in 8 years. If you invest \$1000 at this rate of interest, how much money will you have
a) in 5 years?
b) in 10 years?

2. If money doubles at a certain rate of interest compounded daily in 6 years, how long will it take for the same amount of money to triple in value?

3. Draw a graph showing the time needed to double your money at rate $i^{(1)}$ for the rates 2%, 4%, 6%, …, 20%. Calculate the exact answers and the answers using the Rule of 70.

4. Determine how long \$1 must be left to accumulate at $i^{(12)} = 4.5\%$ for it to amount to twice the accumulated value of another \$1 deposited at the same time at $i^{(2)} = 2.5\%$.

5. Money doubles in t years at rate of interest $i^{(1)}$. At what rate of interest, i^*, will money double in $\frac{t}{2}$ years?

6. You deposit \$800 in an account paying $i^{(2)} = 9\%$, and \$600 in a second account paying $i^{(2)} = 7\%$. When will the first account have twice the accumulated value of the second account?

7. Account A starts now with $100 and pays $i^{(1)} = 4\%$. After 2 years an additional $25 is deposited in account A. Account B is opened 1 year from now with a deposit of $95 and pays $i^{(1)} = 8\%$. When (in years and days) will Account B have $1\frac{1}{2}$ times the accumulated value in Account A if simple interest is allowed for part of a year?

8. In how many years and days should a single payment of $2000 be made in order to be equivalent to payments of $500 now and $800 in 3 years if interest is $i^{(1)} = 8\%$ and simple interest is allowed for part of a year?

9. Bradley puts $100 into Fund X and $100 into Fund Y. Fund Y earns compound interest at the annual rate of $i > 0$, and X earns simple interest at the annual rate of $1.05i$. At the end of 2 years, the amount in Fund Y equals the amount in Fund X. Calculate the amount in Fund Y at the end of 5 years.

10. Jeffrey borrows $1000 from Shirley at an annual effective interest rate i. He agrees to pay back $1000 after 6 years and $1366.87 after 12 years. At time $t = 9$, Jeffrey repays the outstanding balance instead. What is the amount of this payment?

11. Jeffrey deposits $10 into a fund at time $t = 0$ and $20 at time $t = 15$ years. Interest is credited at a nominal discount rate of d compounded quarterly for the first 10 years, and a nominal interest rate of 6% compounded semi-annually thereafter. The accumulated balance in the fund at the end of 30 years is $100. Calculate $d^{(4)}$.

12. The force of interest is $\delta_t = 0.02t$, where t is the number of years from January 1, 2011. If $1 is invested on January 1, 2013, how much is in the fund on January 1, 2018?

13. The accumulated value of $1 at time t, $0 \leq t \leq 1$ is given by a second-degree polynomial in t. You are given that the nominal rate of interest convertible semi-annually for the first half of the year is 5% per annum, and the effective rate of interest for the year is 4% per annum. Calculate δ_t at $t = \frac{3}{4}$.

14. The present value of $1000 due in $2n$ years plus the present value of $2000 due in $4n$ years is $1388.68.

a) What is the value of n if interest is being earned at a simple interest rate of 9.60% and the focal date used is time 0?

b) What is the value of n if interest is being earned at a simple interest rate of 9.60% and the focal date used is time $4n$?

c) What is the value of n if interest is being earned at a simple discount rate of 9.60% and the focal date used is time 0.

Section 2.3 Other Applications of Compound Interest Theory, Inflation, and the "Real" Rate of Interest

We know that the more money you invest at some given interest rate, i, the more dollars of interest you will earn. Further, once you earn a dollar's worth of interest, it becomes a part of the invested money and earns interest itself. The latter characteristic is referred to as multiplicative or geometric growth and is what differentiates compound interest from simple interest.

In any problem where there is geometric growth, even in nonfinancial situations, we can use the theory of compound interest.

EXAMPLE 1 A tree, measured in 2007, contains an estimated 150 cubic metres of wood. If the tree grows at a rate of 3% per annum, how much wood would it contain in 2017?

Solution This is just geometric growth, so we can use compound interest theory. We have $P = 150$, $i = 0.03$, and $n = 10$. Thus, in 2017,

$$\begin{aligned} \text{Amount of wood, } S &= 150(1.03)^{10} \\ &\doteq 202 \text{ cubic metres} \end{aligned}$$

EXAMPLE 2 The population of Canada in July 1997 was 30.3 million people. In July 2007 it was 33.4 million people.

a) What was the annual growth rate from July 1997 to July 2007?
b) At this rate of growth, when will the population reach 40 million people?

Solution a We have $P = 30.3$, $S = 33.4$, $n = 10$, and we wish to determine $i = i^{(1)}$. Hence,

$$\begin{aligned} 30.3(1+i)^{10} &= 33.4 \\ (1+i)^{10} &= \tfrac{33.4}{30.3} \\ 1+i &= (\tfrac{33.4}{30.3})^{1/10} \\ 1+i &= 1.00978841 \\ i &\doteq 0.98\% \end{aligned}$$

Solution b We have $i = 0.0098$ from a), $P = 33.4$, $S = 40$, and we wish to determine n. Therefore,

$$\begin{aligned} 33.4(1.0098)^n &= 40 \\ (1.0098)^n &= \tfrac{40}{33.4} \\ n\log(1.0098) &= \log\tfrac{40}{33.4} \\ n &\doteq 18.49 \text{ years} \\ &\doteq 18 \text{ years } 6 \text{ months} \end{aligned}$$

Therefore the population will reach 40 million sometime in January 2026 assuming the same rate of growth.

EXAMPLE 3 In 1980, Nolan Ryan, a pitcher for the Houston Astros, became the first million-dollar-a-year major league baseball player. In 2001, Alex Rodriguez signed a contract with the Texas Rangers that paid him an average of $25.2 million per season.

a) What is the annual rate of salary inflation from 1980 to 2001?
b) In 2008, Rodriguez had his contract renegotiated so that he earned an average of $28.0 million a year with the New York Yankees. What was the annual rate of growth in his salary from 2001 to 2008?
c) If this growth continues, what would be his expected salary in 2013?

Solution a We have $P = 1\ 000\ 000$, $S = 25\ 200\ 000$, $n = 2001 - 1980 = 21$ years. Thus,

$$\begin{aligned} 1\ 000\ 000(1+i)^{21} &= 25\ 200\ 000 \\ (1+i)^{21} &= 25.2 \\ 1+i &= (25.2)^{1/21} \\ 1+i &= 1.166093459 \\ i &= 16.61\% \end{aligned}$$

Solution b We have $P = 25.2$, $S = 28.0$, $n = 7$
Thus,

$$\begin{aligned} 25.2(1+i)^{7} &= 28.0 \\ (1+i)^{7} &= 28.0/25.2 = 1.1111111 \\ (1+i) &= (1.1111111)^{1/7} \\ 1+i &= 1.015165347 \\ i &= 1.52\% \end{aligned}$$

Solution c In the 2013 season, Rodriguez should expect to re-sign for:

$$S = 28.0(1.015165347)^{5} = \$30.2 \text{ million}$$

Inflation and the "Real" Rate of Interest

One very valuable use of compound interest theory is the analysis of rates of inflation. A widely used measure of inflation is the annual change in the Consumer Price Index. It measures the annual effective rate of change in the cost of a specified "basket" of consumer items.*

EXAMPLE 4 In June 2002 the Consumer Price Index was set at 100. In November 2008 the index was 113.00. That means that if goods cost an average of \$100 in June 2002, they cost \$113.00 in November 2008.

a) Over that 6-year, 5-month period, what was the average annual compound percentage rate of change?
b) If that rate of inflation were to continue, how long would it take before the purchasing power of a June 2002 dollar was only 80¢?

*The following Web site describes the CPI and also includes an inflation calculator. **www.bankofcanada.ca/en/rates/inflation_calc.html**

Solution a We have $P = 100$, $S = 113.00$, $n = 6\frac{5}{12}$ and we wish to determine $i = i^{(1)}$ = rate of inflation. Hence,

$$\begin{aligned} 100(1 + i)^{6\frac{5}{12}} &= 113.00 \\ (1 + i)^{6\frac{5}{12}} &= 1.1300 \\ i &= (1.1300)^{\frac{1}{6.41666}} - 1 \\ &= 1.92\% \end{aligned}$$

Solution b In order to use the fundamental compound interest formula, we set $P = 0.80$, $S = 1$, $i = 0.0192$ and solve for n.

$$\begin{aligned} 0.80(1.0192)^n &= 1 \\ (1.0192)^n &= \frac{1}{0.80} \\ n \log 1.0192 &= -\log 0.8 \\ n &\doteq 11.73 \text{ years} \end{aligned}$$

The last example illustrated the effect of the rate of inflation as measured by the Consumer Price Index (CPI). Inflation rates vary from country to country and from time to time and can be relatively difficult to predict very far into the future. It is also possible to experience a period of time where prices drop and the CPI decreases. This is called deflation.

There is often a relationship between interest rates and rates of inflation. To illustrate this point, let's suppose that today you have \$100 and you could buy a basket of goods with that \$100. Suppose there is an inflation rate of 5%. This same basket of goods will now cost \$105 at the end of the year. If you have not invested your \$100, then at the end of the year it would be worth only \$100/(1.05) = \$95.24. In other words, if you wanted to buy that same basket of goods, you would be able to afford only 95.24% of the goods in the basket and would have to throw out 4.76% of them.

Had you invested your \$100 at $i = 5\%$, you would have had \$105 at the end of the year and would have been able to buy the exact same basket of goods. And if you had earned more than 5%, you would have been ahead of the game at the end of the year. (For example, if you earned 8%, you would have had \$108 at the end of the year and it would have bought \$108/1.05 = \$102.86 worth of goods, an increase of 2.86%.)

Investors need to take into account the rate of inflation when calculating the rate of return they have earned on an investment. When doing so, investors are calculating what is referred to as the "real" rate of return.

If i is the annual rate of interest being paid in the marketplace and r is the annual rate of inflation, then \$1 invested at the beginning of the year will grow to $\$(1 + i)$ at the end of the year. However, its purchasing power is equal to only $\$\left(\frac{1 + i}{1 + r}\right)$. Hence, the real rate of return is

$$\boxed{i_{real} = \frac{1 + i}{1 + r} - 1 = \frac{i - r}{1 + r}} \qquad \textbf{(10)}$$