ACTEX Academic Series

Financial Mathematics
Theory & Practice

Second Edition

2nd

Robert Brown
Steve Kopp

Robert L. Brown

Steve Kopp

FINANCIAL MATHEMATICS:

Theory and Practice

2nd Edition

ACTEX Learning

Request for permission should be addressed to:
ACTEX Learning
PO Box 69
Greenland, NH 03840
support@actexlearning.com

Manufactured in the United States of America

Cover Design by Rachel Maragnano

Table of Contents

Preface

Welcome to an alternative approach to the teaching and learning of financial mathematics!

We have a combined 80 years of teaching experience in actuarial science in general, and financial mathematics in particular. (Robert is now retired from the University of Waterloo, where he taught actuarial science for 40 years, and Steve is currently in his 40th year of teaching actuarial science and statistics at the University of Western Ontario.) In those 80 years, we have always lamented the fact that while there were multiple references available to our students, we had to spend extra time in preparation of our classes and assignments to fill in certain pedagogical deficiencies in any one of these references. Thus, our desire to enter the textbook listings for this discipline.

Our textbook is designed for mathematics / actuarial students with a strong grounding in, and understanding of, a basic first-year calculus curriculum. Our target audience contains students who plan to write the established actuarial examinations as made available around the world and, some day, to have a career in the actuarial or finance professions.

This textbook is designed to provide the reader with a generic approach to understanding financial mathematics with respect to a wide range of financial transactions, including annuities, mortgages, personal loans, and bonds, as well as a limited knowledge of advanced topics such as duration and immunization. All examples are solved from absolute first principles and we propose that any student should solve all financial mathematics problems in the same way; that is, from absolute first principles.

As previously stated, this textbook assumes a knowledge of introductory calculus, so problems are posed whose solutions may require the use of derivatives, or integrals, or both. A comfort with exponents and logarithms is also assumed.

We would be remiss if we did not thank Yijia Liu, Marta Santos and Yongkang Wang at ACTEX Learnnig, for their essential assistance in creating this textbook.

This textbook acknowledges the existence of modern financial calculators and spreadsheets and uses these technologies where appropriate. However, in each and every instance, our goal is to have the student be equipped to understand each solution from fundamental principles.

Our passion for, and interest in, financial mathematics remains undiminished. It is our sincere hope that this textbook provides the student with an improved understanding of the fascinating calculations that underlie most financial transactions.

Robert Brown, Etobicoke, Ontario

Steve Kopp, London, Ontario

2

Equivalence Equations

Learning Objectives

In Chapter 2, we learned how to perform financial calculations using simple and compound interest and using simple and compound discount. We also worked with nominal rates of interest and discount. Most financial transactions involve compound interest rates. Beginning with this chapter, the rest of this textbook is devoted to financial calculations using compound interest, unless stated otherwise.

Upon completing this chapter, you will be able to do the following:

- Calculate equivalent debts and payments using an equation of value.
- Determine how long it will take to earn a certain amount of interest.
- Calculate the rate of return on an investment.
- Solve non-financial problems that involve geometric growth.
- Calculate the price of Treasury bills in both Canada and the United States.

2.1 Equations of Value

All financial decisions must take into account the basic idea that **money has time value**. In other words, receiving $100 today is not the same as receiving $100 one year ago, nor receiving $100 one year from now if there is a positive interest rate. In a financial transaction involving money due on different dates, every sum of money should have an attached date, the date on which it falls due. That is, we need to deal with **dated values**. This is one of the most important facts in the mathematics of finance.

Illustration: At an annual effective interest rate of 8%, \$100 due in one year is considered to be equivalent to \$108 in two years since \$100 would accumulate to \$108 in one year. In the same way,

$$\$100(1+0.08)^{-1} = \$92.59$$

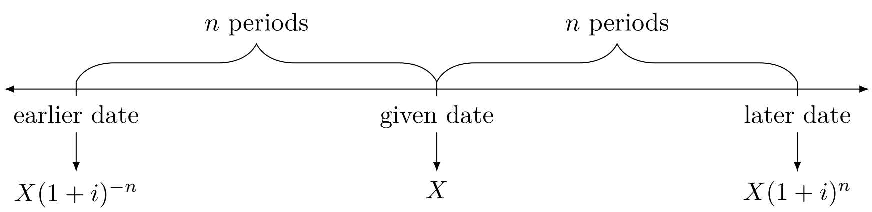
would be considered an equivalent sum today.

Another way to look at this is to suppose you were offered \$92.59 today, or \$100 one year from now, or \$108 two years from now. All three values are considered dated values. Which one would you choose? Most people would probably take the \$92.59 today because it is money in their hands. However, from a financial point of view, all three values are the same, or are **equivalent**, since you could take the \$92.59 and invest it for one year at 8%, after which you would have \$100. This \$100 could then be invested for one more year at 8% after which it would have accumulated to \$108. Note that the three dated values are not equivalent at some other rate of interest.

In general, we compare dated values by the following **definition of equivalence**:

> X due on a given date is equivalent at a given compound interest rate i to Y due n periods later, if $Y = X(1+i)^n$ or $X = Y(1+i)^{-n}$

The following diagram illustrates dated values equivalent to a given dated value X.



Based on the time diagram above we can state the following simple rules:

> 1. When we move money forward in time, we **accumulate**, i.e., we multiply the sum by an accumulation factor $(1+i)^n$.
> 2. When we move backward in time, we **discount**, i.e., we multiply the sum by a discount factor $(1+i)^{-n}$.

The following property of equivalent dated values, called the *property of transitivity*, holds under compound interest:

At a given compound interest rate, if X is equivalent to Y, and Y is equivalent to Z, then X is equivalent to Z.[1]

To prove this property we arrange our data on a time diagram.

If X is equivalent to Y, then $Y = X(1+i)^{n_2-n_1}$
If Y is equivalent to Z, then $Z = Y(1+i)^{n_3-n_2}$

Eliminating Y from the second equation we obtain:

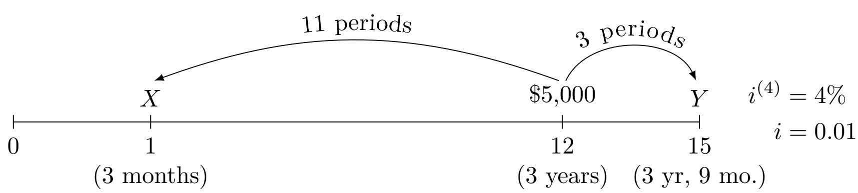
$$Z = X(1+i)^{n_2-n_1}(1+i)^{n_3-n_2} = X(1+i)^{n_3-n_1}$$

Thus Z is equivalent to X.

Example 2.1. A debt of $5,000 is due at the end of 3 years. Determine an equivalent debt due at the end of a) 3 months; b) 3 years 9 months, if $i = 4\%$.

SOLUTION

We arrange the data on a time diagram below.



By definition of equivalence:

$$\text{a) } X = \$5{,}000(1.01)^{-11} = \$4{,}481.62$$
$$\text{b) } Y = \$5{,}000(1.01)^{3} = \$5{,}151.51$$

Note that X and Y are equivalent by verifying

$$Y = X(1.01)^{14} \text{ or } \$5{,}151.51 = \$4{,}481.62(1.01)^{14}.$$

□

[1]The property of transitivity does not hold if simple interest is involved. In the initial example of this section, if the 8% was a simple interest rate, $92.59 one year ago is equivalent to $100 today and $100 today is equivalent to $108 one year from today. But the $92.59 is not equivalent to $108.

The sum of a set of dated values, due on different dates, has no meaning. We have to replace all the dated values by equivalent dated values, due on the same date. The sum of the equivalent values is called the **dated value of the set**.

At compound interest the following property is true: *The various dated values of the same set are equivalent.*

Example 2.2. A person owes \$200 due in 6 months and \$300 due in 15 months. What single payment a) now, b) in 12 months, will liquidate these obligations if money is worth $i^{(12)} = 6\%$?

SOLUTION

We arrange the data on the diagram below. Let X be the single payment due now and Y be the single payment due in 12 months.

a) To calculate the equivalent dated value X, we must discount the \$200 debt for 6 months and discount the \$300 debt for 15 months.

$$X = \$200(1.005)^{-6} + \$300(1.005)^{-15} = \$194.10 + \$278.38 = \$472.48$$

b) To calculate the equivalent dated value Y, we must accumulate the \$200 debt from time 6 to time 12, or 6 periods, and discount the \$300 debt from time 15 to time 12, or 3 periods.

$$Y = \$200(1.005)^{6} + \$300(1.005)^{-3} = \$206.08 + \$295.54 = \$501.62$$

We can verify the property of transitivity of X and Y by showing that

$$Y = X(1.005)^{12} = \$472.48(1.005)^{12} = \$501.62,$$

or

$$X = Y(1.005)^{-12} = \$501.62(1.005)^{-12} = \$472.48.$$

□

One of the most important problems in the mathematics of finance is the replacing of a given set of payments by an equivalent set. Two sets of payments are equivalent at a given compound interest rate if the dated values of the sets, on any common date, are equal. An equation stating that the dated values, on a common date, of two sets of payments are equal is called an **equation of value** or an **equation of equivalence**. The date used is called the **focal date** or the **comparison date** or the **valuation date**.

The procedure for setting up and solving an equation of value is stated below.

Step 1 Draw a good time diagram showing the dated values of debts on one side of the time line and the dated values of payments on the other side. The times on the line should be stated in terms of interest compounding periods.

Step 2 Select one, and only one, focal date and accumulate/discount all dated values to the focal date using the specified compound interest rate.

Step 3 Set up an equation of value at the focal date:

dated value of payments = dated value of debts.

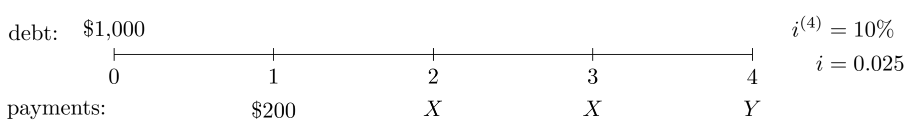
Step 4 Solve the equation of value using methods of algebra.

The following examples illustrate the use of equations of value in the mathematics of finance.

Example 2.3. A debt of \$1,000 with interest at $i^{(4)} = 10\%$ will be repaid by a payment of \$200 at the end of 3 months and three equal payments at the ends of 6, 9, and 12 months. What will these payments be?

SOLUTION

We arrange all the dated values on a time diagram.



Any date can be selected as a focal date. We show the calculation using the end of 12 months and the present time.

Equation of value at the end of 12 months:

$$\begin{aligned}\text{dated value of the payments} &= \text{dated value of the debts}\\ 200(1.025)^3 + X(1.025)^2 + X(1.025)^1 + X &= 1{,}000(1.025)^4\\ 215.38 + 1.050625X + 1.025X + X &= 1{,}103.81\\ 3.075625X &= 888.43\\ X &\doteq 288.86\end{aligned}$$

Equation of value at the present time:

$$\begin{aligned}200(1.025) - 1 + X(1.025) - 2 + X(1.025) - 3 + X(1.025) - 4 &= 1{,}000.00\\ 195.12 + 0.951814396X + 0.928599411X + 0.905950645X &= 1{,}000.00\\ 2.786364452X &= 804.88\\ X &\doteq 288.86\end{aligned}$$

□

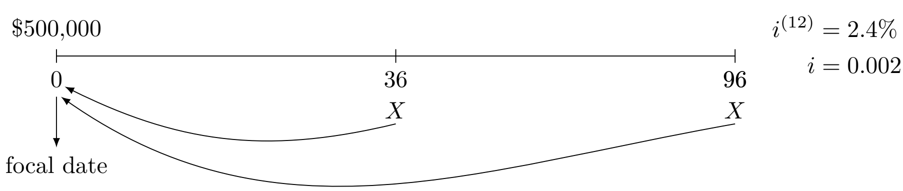
Calculation Tip

Choose a convenient focal date, one that simplifies your calculations, when using equations of value at compound interest.

Example 2.4. A man leaves an estate of \$500,000 that is invested at $i^{(12)} = 2.4\%$. At the time of his death, he has two children aged 13 and 18. Each child is to receive an equal amount from the estate when they reach age 21. How much does each child get?

SOLUTION

The older child will get $\$X$ in 3 years (36 months); the younger child will get $\$X$ in 8 years (96 months). We arrange the dated values on a time diagram.



Equation of value at present:

$$\begin{aligned} X(1.002)^{-36} + X(1.002)^{-96} &= 500{,}000 \\ 0.930597807X + 0.825465131X &= 500{,}000 \\ 1.756062939X &= 500{,}000 \\ X &\doteq \$284{,}727.84 \end{aligned}$$

Each child will receive \$284,727.84

The following calculation checks the correctness of the answer:

$$\begin{aligned} \text{Amount in fund at the end of 3 years} &= \$500{,}000(1.002)^{36} = \$537{,}289.04 \\ \text{Payment to the older child} &= \$284{,}727.84 \\ \text{Balance in the fund} &= \$252{,}561.20 \\ \text{Amount in fund after 5 more years} &= \$252{,}561.20(1.002)^{60} = \$284{,}727.83 \end{aligned}$$

The 1-cent difference is due to rounding.

□

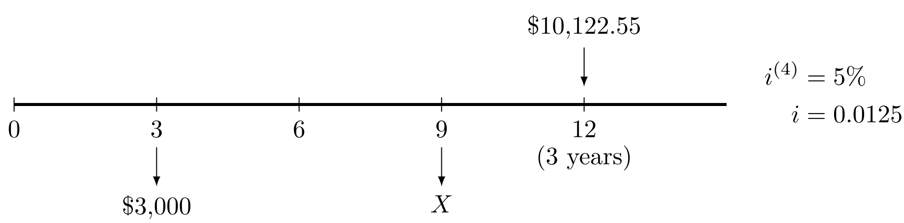
Example 2.5. Stephanie owes \$8,000 due at the end of 3 years with interest at $i^{(2)} = 8\%$. The lender agrees to allow her to pay back the loan early with a payment of \$3,000 at the end of 9 months and X at the end of 27 months. If the lender can reinvest any payments at $i^{(4)} = 5\%$, what is X?

SOLUTION

In this example, there are two interest rates: 8% is the interest rate on the original loan, while 5% is the rate that will be used in the equation of value. First we need to calculate the maturity value of the loan, which is due in 3 years (6 half-years):

$$S = 8{,}000(1.04)^6 = 10{,}122.55$$

We now set up our time diagram, with \$10,122.55 as the original debt and using quarter-years as the times.



Equation of value at time 9:

$$\begin{aligned} 3{,}000(1.0125)6 + X &= 10{,}122.55(1.0125)^{-3} \\ 3{,}232.149542 + X &= 9{,}752.250203 \\ X &\doteq 6{,}520.10 \end{aligned}$$

A payment of \$3,000 at the end of 9 months and \$6,520.10 at the end of 27 months is equivalent to a debt of \$10,122.55 at the end of 3 years. To check this is the correct answer:

Payment at end of 9 month, invested at $i^{(4)} = 5\%, \$3{,}000(1.0125)^6 = \$3{,}232.15$

Payment at end of 27 months = \$6,520.10

Balance \$9,752.25

Balance invested at $i^{(4)} = 5\%$
to end of 3 years $= \$9{,}752.25(1.0125)^3 = \$10{,}122.55$
which is equal to the original debt owed at the end of 3 years.

□

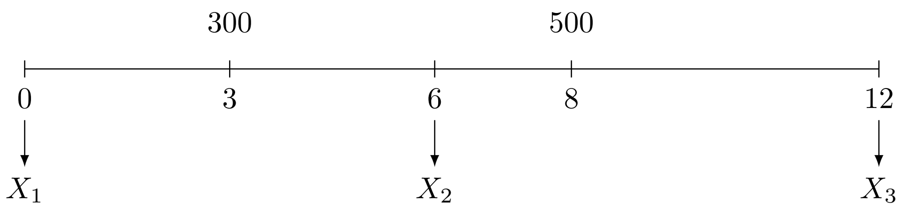
An equation of value can be set up for problems involving simple interest or simple discount. The procedure is exactly the same, except the choice of focal date will change your final answer. That is because the property of transitivity does not hold when using simple interest or simple discount accumulation functions or discount functions. Thus, both parties to a financial transaction must agree on which focal date to use.

Observation

In mathematics, an equivalence relation must satisfy the property of transitivity. Thus, strictly speaking, the equivalence of dated values at a simple interest rate is not an equivalence relation, and that is why when using simple interest you will get a different answer depending on what focal date is used. However, the choice of focal date under compound interest will not affect the final answer because the equivalence of dated values at a compound interest rate is an equivalence relation.

Example 2.6. A person owes \$300 due in 3 months and \$500 due in 8 months. The lender agrees to allow the person to pay off these two debts with a single payment. What single payment a) now; b) in 6 months; c) in 1 year will liquidate these obligations if money is worth 8% simple interest?

SOLUTION



We calculate equivalent dated values of both obligations using the three different focal dates and arrange them in the table below.

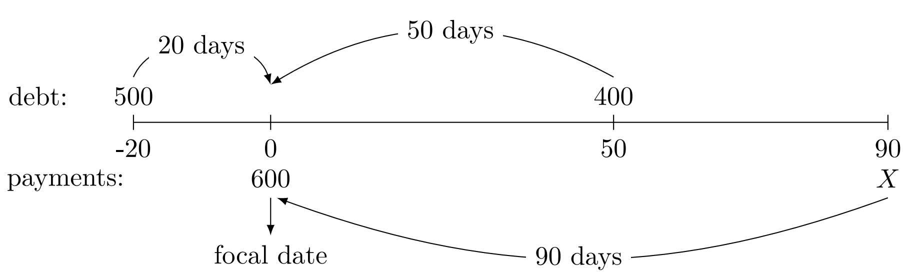
Obligations	**Now**	**In 6 Months**	**In 1 Year**
First	$300\left[1+(0.08)\left(\frac{3}{12}\right)\right]^{-1}$ $= 294.12$	$300\left[1+(0.08)\left(\frac{3}{12}\right)\right]$ $= 306.00$	$300\left[1+(0.08)\left(\frac{9}{12}\right)\right]$ $= 318.00$
Second	$500\left[1+(0.08)\left(\frac{8}{12}\right)\right]^{-1}$ $= 474.68$	$500\left[1+(0.08)\left(\frac{2}{12}\right)\right]^{-1}$ $= 493.42$	$500\left[1+(0.08)\left(\frac{4}{12}\right)\right]$ $= 513.33$
Sum	$X_1 = 768.80$	$X_2 = 799.42$	$X_3 = 831.33$

□

Example 2.7. Debts of \$500 due 20 days ago and \$400 due in 50 days are to be settled by a payment of \$600 now and a final payment 90 days from now. Determine the value of the final payment at a simple interest rate of 11% with a focal date at the present.

SOLUTION

We arrange the dated values on a time diagram.



Equation of value at the present time:

$$\begin{aligned}
\text{dated value of payments} &= \text{dated value of debts} \\
X\left[1+(0.11)\left(\frac{90}{365}\right)\right]^{-1}+600 &= 500\left[1+(0.11)\left(\frac{20}{365}\right)\right] \\
&\quad +400\left[1+(0.11)\left(\frac{50}{365}\right)\right]^{-1} \\
0.973592958X+600 &= 503.01+394.06 \\
0.973592958X &= 297.07 \\
X &\doteq 305.13
\end{aligned}$$

The final payment to be made in 90 days is \$305.13.

□

Exercise 2.1

Part A

1. An obligation of \$2,500 falls due at the end of 7 years. Determine an equivalent debt at the end of a) 3 years and b) 10 years, if $i^{(12)} = 9\%$.

2. One thousand dollars is due at the end of 2 years and \$1,500 at the end of 4 years. If money is worth $i^{(4)} = 8\%$, calculate an equivalent single amount at the end of 3 years.

3. Eight hundred dollars is due at the end of 4 years and \$700 at the end of 8 years. If money is worth $i^{(12)} = 3\%$, determine an equivalent single amount at
   a) the end of 2 years;
   b) the end of 6 years;
   c) the end of 10 years;

   Show your answers are equivalent.

4. A debt of \$2,000 is due at the end of 8 years. If \$1,000 is paid at the end of 3 years, what single payment at the end of 7 years would liquidate the debt if money is worth $i^{(2)} = 7\%$?

5. A person borrows \$4,000 at $i^{(4)} = 6\%$. He promises to pay \$1,000 at the end of one year, \$2,000 at the end of 2 years, and the balance at the end of 3 years. What will the final payment be?

6. A consumer buys goods worth \$1,500. She pays \$300 down and will pay \$500 at the end of 6 months. If the store charges $i^{(12)} = 18\%$ on the unpaid balance, what final payment will be necessary at the end of one year?

7. A debt of \$1,000 is due at the end of 4 years. If money is worth $i^{(12)} = 3.6\%$, and \$375 is paid at the end of 1 year, what equal payments at the end of 2 and 3 years respectively would liquidate the debt?

8. On September 1, 2023, Paul borrowed \$3,000, agreeing to pay interest at 12% compounded quarterly. He paid \$900 on March 1, 2024, and \$1,200 on December 1, 2024.
   a) What equal payments on June 1, 2025, and December 1, 2025, will be needed to settle the debt?
   b) If Paul paid \$900 on March 1, 2024, \$1,200 on December 1, 2024, and \$900 on March 1, 2025, what would be his outstanding balance on September 1, 2025?

9. A woman's bank account showed the following deposits and withdrawals:

	Deposits	**Withdrawals**
January 1, 2024	\$200	
July 1, 2024	\$150	
January 1, 2025		\$250
July 1, 2025	\$100	

If the account earns $i^{(2)} = 6\%$, determine the balance in the account on January 1, 2026.

10. Instead of paying \$400 at the end of 5 years and \$300 at the end of 10 years, a man agrees to pay $\$X$ at the end of 3 years and $\$2X$ at the end of 6 years. Determine X if $i^{(1)} = 10\%$.

11. A man stipulates in his will that \$50,000 from his estate is to be placed in a fund from which his three children are to each receive an equal amount when they reach age 21. When the man dies, the children are ages 19, 15, and 13. If this fund earns interest at $i^{(2)} = 6\%$, how much does each receive?

12. To pay off a loan of \$4,000 at $i^{(12)} = 7.2\%$, Ms. Fil agrees to make three payments in 3, 7, and 12 months respectively. The second and third payment are to be double the first. What is the size of the first payment?

13. Mrs. Singh borrows \$3,000, due with interest at $i^{(12)} = 9\%$ in 2 years. The lender agrees to let Mrs. Singh repay the loan with a payment of \$1,000 in 6 months, \$1,500 in 12 months, and $\$X$ in 30 months. If money is worth $i^{(2)} = 6\%$, what is the value of X?

14. Therèse owes \$500 due in 4 months and \$700 due in 9 months. If money is worth 7% simple interest, what single payment

a) now;

b) in 6 months;

c) in 1 year;

will liquidate these obligations?

15. Andrew owes Nicola \$500 in 3 months and \$200 in 6 months both due with simple interest at 6%. If Nicola accepts \$300 now, how much will Andrew be required to repay at the end of 1 year, provided they agree to use a simple interest rate of 8% and a focal date at the end of 1 year?

16. A person borrows \$1,000 to be repaid with two equal instalments, one in 6 months, the other at the end of 1 year. What will be the size of these payments if the simple discount rate is $d = 8\%$ and the focal date is 1 year hence? What if the focal date is today?

17. Mrs. Adams has two options available in repaying a loan. She can pay \$200 at the end of 5 months and \$300 at the end of 10 months, or she can pay $\$X$ at the end of 3 months and $\$2X$ at the end of 6 months. Determine X if simple interest is at 12% and the focal date is 6 months hence and the options are equivalent. What is the answer if the focal date is 3 months hence and the options are equivalent?

18. A debt of \$5,000 is due at the end of 5 years. It is proposed that $\$X$ be paid now with another $\$X$ paid in 10 years time to liquidate the debt. Calculate the value of X if the effective interest rate is 6.1% for the first 6 years and 4.8% for the next 4 years.

19. A company wishes to replace the following three debts:

\$20,000 due on July 1, 2023
\$30,000 due on January 1, 2026 and
\$35,000 due on July 1, 2029

with a single debt of $\$Y$ payable on January 1, 2026. Calculate the value of Y if $i^{(2)} = 12\%$ prior to January 1, 2026 and $i^{(2)} = 10\%$ after January 1, 2026.

20. (SOA) A financial institution agrees to lend \$12,000 now and another $\$X$ at the end of 2 years in exchange for a payment of \$65,000 at the end of 8 years. The institution charges an annual effective rate of 6% for the first 5 years followed by a force of interest $\delta_t = \frac{1}{t+1}$, $t \geq 5$ thereafter. What is the value of X?

Part B

1. If money is worth $i^{(1)} = 8\%$, what single sum of money payable at the end of 2 years will equitably replace \$1,000 due today plus a \$2,000 debt due at the end of 4 years with interest at $12\frac{1}{2}\%$ per annum compounded semi-annually?

2. On January 1, 2022, Mr. Planz borrowed \$5,000 to be repaid in a lump-sum payment with interest at $i^{(4)} = 9\%$ on January 1, 2028. It is now January 1, 2024. Mr. Planz would like to pay \$500 today and complete the liquidation with equal payments on January 1, 2026, and January 1, 2028. If money is now worth $i^{(4)} = 8\%$, what will these payments be?

3. You are given two loans, with each loan to be repaid by a single payment in the future. Each payment will include both principal and interest. The first loan is repaid by a \$3,000 payment at the end of 4 years. The interest is accrued at $i^{(2)} = 10\%$. The second loan is repaid by a \$4,000 payment at the end of 5 years. The interest is accrued at $i^{(2)} = 8\%$. These two loans are to be consolidated. The consolidated loan is to be repaid by two equal instalments of $\$X$, with interest at $i^{(2)} = 6\%$. The first payment is due immediately and the second payment is due one year from now. Calculate X.

4. You are given the following data on three series of payments:

	Payment at End of Year			**Accumulated Value at End of Year 18**
	6	12	18	
Series A	240	200	300	X
Series B	0	360	700	$X + 100$
Series C	Y	600	0	X

Assume interest is compounded annually. Calculate Y.

2.2 Determining the Rate and Time

Determining the Rate: When S, P, and n are given, we can substitute the given values into the fundamental **compound interest formula** $S = P(1+i)^n$ and solve it for the unknown interest rate i.

$$S = P(1+i)^n$$
$$(1+i)^n = \frac{S}{P}$$
$$1 + i = \left(\frac{S}{P}\right)^{1/n}$$

$$\boxed{i = \left(\frac{S}{P}\right)^{1/n} - 1} \qquad (2.1)$$

Calculation Tip

Using the power key of our calculator, we calculate the exact value of i. The nominal annual rate of interest is determined by multiplying the effective rate i by the number of compounding periods per year, $i^{(m)} = mi$, and then rounding off to the nearest hundredth of a percent.

Example 2.8. At what nominal rate $i^{(12)}$ will money triple itself in 12 years?

SOLUTION

We can use any sum of money as the principal. Let $P = x$, then $S = 3x$, and $n = 12 \times 12 = 144$.

Substituting in $S = P(1+i)^n$ we obtain an equation for the unknown interest rate i per month:

$$3x = x(1+i)^{144}$$
$$(1+i)^{144} = 3$$

Solving the exponential equation $(1+i)^{144} = 3$ directly for i and then using a pocket calculator we have:

$$\begin{aligned} (1+i)^{144} &= 3 \\ 1+i &= 3^{1/144} \\ i &= 3^{1/144} - 1 \\ i &= 0.007658429 \\ i^{(12)} &= 12i = 0.091901147 \\ i^{(12)} &\doteq 9.19\% \end{aligned}$$

□

Using a Financial Calculator

Many of the calculations that have been presented so far in this textbook can be performed using a financial **calculator**.

However, for most of the exercises, it is easiest to set up an equation of value and solve for the answer using the symbols and formulas presented in this text, using a calculator to do only the basic math. There has been no real need for a financial calculator. However, there are situations where a financial calculator can lead to quicker answers, and calculating the value of i is one such situation.

There are many different financial calculators on the market. Four good ones are: Sharp EL-733A, Hewlett Packard 10B, Texas Instruments BA-35, and Texas Instruments BA-II Plus. We will illustrate the calculations using the notation from the Texas Instruments BA-II Plus calculator.

In Example 2.8, we have $PV = 1$ (note that you need a negative sign in front of the present value), $FV = S = 3$, and $n = 144$ (the term is always entered as the number of interest periods). The steps are as follows:

-1 [PV] 3 [FV] 144 [N] [CPT] [I/Y]

0.765842888 (per month)

Note that the answer is given as a percentage. That is, the monthly rate is 0.765842888%. To obtain the nominal rate, we need to multiply by 12:

$$i^{(12)} = 12(0.76584288)\% = 9.190114656\% \doteq 9.19\%$$

Example 2.9. Suppose you would like to earn \$600 of interest over 21 months on an initial investment of \$2,500. What nominal rate of interest, $i^{(4)}$, is needed?

SOLUTION

We have $P = 2{,}500$, $S = 2{,}500 + 600 = 3{,}100$, $n = 1.75 \times 4 = 7$ quarters, and $i = i^{(4)}/4$. Setting up the equation and solving, we get

$$\begin{aligned} 3{,}100 &= 2{,}500(1+i)^7 \\ 1 + i &= (1.24)^{1/7} \\ i &= (1.24)^{1/7} - 1 \\ i &= 0.031207244 \\ i^{(4)} &= 4i = 0.124828974 \\ i^{(4)} &\doteq 12.48 \end{aligned}$$

Using the BA-II Plus calculator:

−2,500 [PV] 3,100 [FV] 7 [N] [CPT] [I/Y]

3.120724362 (per quarter)

To obtain the nominal rate:

$$i^{(4)} = 4(3.120724362)\% = 12.482897\% \doteq 12.48\%$$

□

Determining the Time: When S, P, and i are given, we can substitute the given values into the fundamental compound amount formula $S = P(1+i)^n$ and solve it for the unknown n using logarithms.

In this textbook we assume that students have pocket calculators with a built-in common logarithmic function $\log x$ and its inverse function 10^x.

Solving the formula $S = P(1+i)^n$ for n we obtain:

$$\begin{aligned} (1+i)^n &= \frac{S}{P} \\ n \log(1+i) &= \log\left(\frac{S}{P}\right) \end{aligned}$$

$$\boxed{n = \frac{\log\left(\frac{S}{P}\right)}{\log(1+i)}} \tag{2.2}$$

Example 2.10. How long will it take \$500 to accumulate to \$850 at $i^{(12)} = 5.4\%$?

SOLUTION

Let n represent the number of months, then we have:

$$\begin{aligned} 500(1.0045)^n &= 850 \\ (1.0045)^n &= \frac{850}{500} \\ (1.0045)^n &= 1.7 \end{aligned}$$

We have:

$$\begin{aligned} n \log 1.0045 &= \log 1.7 \\ n &= \frac{\log 1.7}{\log 1.0045} \\ n &= 118.1825 \text{ months} \\ n &= 9.84854 \text{ years} \\ &= 9 \text{ years}, \quad \overbrace{\underset{= 10.18248}{0.84854 \times 12}}^{10 \text{ months},} \quad \overbrace{\underset{= 5.4744 \doteq 6}{0.18248 \times 30}}^{6 \text{ days}} \end{aligned}$$

ALTERNATIVE SOLUTION

The accumulated value of \$500 for 118 periods at $i^{(12)} = 5.4\%$ is $500(1.0045)^{118} = \$849.30$. Now we calculate how long it will take \$849.30 to accumulate \$0.70 simple interest at rate 5.4%.

$$t = \frac{i}{Pr} = \frac{0.70}{849.30 \times 0.054} = 0.015263114 \text{ years } \doteq 6 \text{ days}$$

Thus the time is 9 years, 10 months, and 6 days. □

Using a Financial Calculator

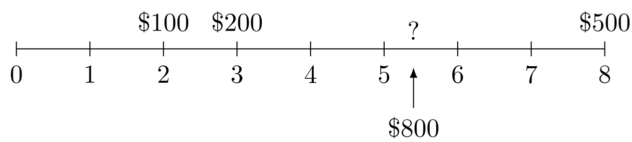
To calculate n using the BA-II Plus calculator, we have $PV = 500$, $FV = 850$, and $I/Y = 0.45$ (the interest rate is the effective rate per compounding period, entered as a number, not a percentage; $5.4/12 = 0.45$). The steps are as follows:

−500 [PV] 850 [FV] 0.45 [I/Y] [CPT] [N]

118.1825047 (months)

Example 2.11. Payments of \$100, \$200, and \$500 are paid at the end of years 2, 3, and 8 respectively. If $i^{(1)} = 5\%$ per annum effective, determine the time, t, at which a single payment of \$800 would be equivalent.

SOLUTION



Using time 0 as the focal date.

$$\begin{aligned} 800(1.05)^{-t} &= 100(1.05)^{-2} + 200(1.05)^{-3} + 500(1.05)^{-8} \\ 800(1.05)^{-t} &= 601.89 \\ (1.05)^{t} &= \frac{800}{601.89} = 1.32915 \\ t &= \frac{\log 1.32915}{\log 1.05} = 5.832 \text{ years} \end{aligned}$$

□

Example 2.12. Jerry invests \$1,000 today in a fund earning $i^{(2)} = 5\%$. How long will it take for his money to double?

SOLUTION

We have $P = 1{,}000$, $S = 2{,}000$, $i = 0.025$ and will let n represent the number of half-years. Setting up the equation, we obtain

$$\begin{aligned} 1{,}000(1.025)^{n} &= 2{,}000 \\ (1.025)^{n} &= 2 \\ n \log 1.025 &= \log 2 \\ n &= \frac{\log 2}{\log 1.025} \\ n &= 28.0710 \text{ half years} \\ &= 14.0355 \text{ years} \\ &= 14 \text{ years}, \underbrace{0 \text{ months},}_{\substack{0.0355 \times 12 \\ = 0.426}} \quad \underbrace{13 \text{ days}}_{\substack{0.4260 \times 30 \\ = 12.78 \doteq 13}} \end{aligned}$$

Note: If the interest rate had been $i^{(2)} = 10\%$, $n = 7$ years, 1 month, 8 days.

Using the BA-II Plus calculator and $i = 0.05/2 = 0.025$ per half-year,

$-1{,}000$ [PV] $2{,}000$ [FV] 2.5 [I/Y] [CPT] [N]

28.07103453 (half-years)

Using $i = 0.10/2 = 0.05$ per half-year,

$-1{,}000$ [PV] $2{,}000$ [FV] 5 [I/Y] [CPT] [N]

14.20669908 (half-year)

□

Rule of 70

A quick estimate for the amount of time needed for money to double in value can be obtained using the "**Rule of 70**." According to this rule, the number of interest periods needed for money to double in value is approximately equal to 70 divided by the effective rate of interest per period.

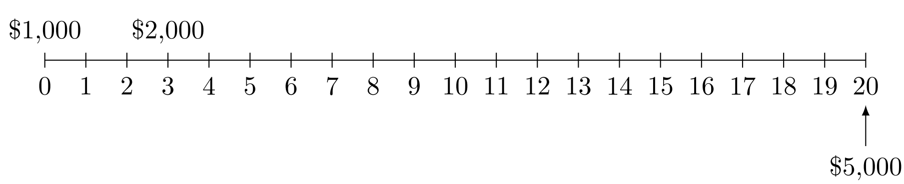
Using this rule in Example 2.12, we obtain,

i) If $i^{(2)} = 5\%$, then $i = 0.025$ and $n \doteq 70/2.5 = 28.0$ half-years. Compare this to the correct answer of 28.0710.

ii) If $i^{(2)} = 10\%$, then $i = 0.05$ and $n \doteq 70/5 = 14.0$ half-years. Compare this to the correct answer of 14.2067.

Example 2.13. At what rate of interest, $i^{(2)}$, would an investment of \$1,000 immediately and \$2,000 in 3 years accumulate to \$5,000 in 10 years?

SOLUTION



Using the end of 10 years (time 20) as the focal date,

$$1{,}000(1+i)^{20} + 2{,}000(1+i)^{14} = 5{,}000.$$

This cannot be solved algebraically. Instead you need to use "trial and error".

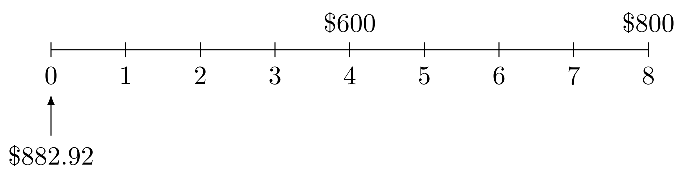
$i = \frac{i^{(2)}}{2}$	**LHS**
0.05	\$6,613.16
0.04	\$5,654.48
0.035	\$5,227.18
0.033	\$5,065.18
0.0322	\$5,001.76
0.03218	\$5,000.18

Rounding to five decimal points should be enough accuracy. Thus, the desired nominal rate of interest, $i^{(2)} = 2(0.03218) = 6.436\%$.

□

Example 2.14. A deposit of \$882.92 today will allow you to withdraw \$600 in 4 years and another \$800 in 8 years. What effective rate of interest is assumed?

SOLUTION



Using time 8 as our focal date:

$$882.92(1+i)^8 = 600(1+i)^4 + 800$$
$$882.92(1+i)^8 - 600(1+i)^4 - 800 = 0$$

This is a quadratic equation in the form $ax^2 + bx + c = 0$, where $x = (1+i)^4$, $a = 882.92$, $b = -600$, and $c = -800$.

Thus:

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{600 \pm \sqrt{(-600)^2 - 4(882.92)(-800)}}{2(882.92)} \\ &= -0.670928965 \text{ or } +1.350492233 \end{aligned}$$

$$
\begin{aligned}
(1+i)^4 &= 1.350492233 \\
i &= (1.350492233)^{\frac{1}{4}} - 1 \\
&= 0.078010579 \\
&= 7.80\%
\end{aligned}
$$

□

Exercise 2.2

Part A

1-4. In problems 1 to 4 calculate the nominal rate of interest.

No.	Principal	Amount	Time	Conversion
1.	$2,000	$3,000	3 years 9 months	quarterly
2.	$ 100	$ 150	4 years 7 months	monthly
3.	$ 200	$ 600	15 years	annually
4.	$1,000	$1,581.72	3 years 6 months	semi-annually

5-8. In problems 5 to 8 determine the time.

No.	Principal	Amount	Interest Rate	Conversion
5.	$2,000	$2,800	4%	quarterly
6.	$ 100	$ 130	9%	semi-annually
7.	$ 500	$ 800	6%	monthly
8.	$1,800	$2,200	8%	weekly

9. An investment fund advertises that it will guarantee to double your money in 10 years. What rate of interest $i^{(1)}$ is implied?

10. If an investment grows 50% in 4 years, what rate of interest $i^{(4)}$ is being earned?

11. From 2023 to 2028, the earnings per share of common stock of a company increased from \$4.71 to \$9.38. What was the compounded annual rate of increase?

12. At what rate $i^{(365)}$ will an investment of \$4,000 grow to \$6,000 in 3 years?

13. How long will it take to double your deposit in a savings account that accumulates at
   a) $i^{(1)} = 4.56\%$?
   b) $i^{(365)} = 7\%$?
   c) Redo a) and b) using the Rule of 70.

14. How long will it take for \$800 to grow to \$1,300 in a fund earning interest at rate 4.9% compounded semi-annually?

15. How long will it take to increase your investment by 50% at rate 5% compounded daily?

16. The present value of \$1,000 due in $2n$ years plus the present value of \$2,000 due in $4n$ years is \$1,388.68. If the interest rate is $i^{(12)} = 9.60\%$, what is the value of n?

17. You need to buy some furniture that costs \$12,900 in cash today. Alternatively, you can make a payment of \$4,429 today, followed by \$4,429 in 3 months, and \$4,429 in 6 months. For the second option, the furniture company claims they are only charging you interest at a rate of $i^{(1)} = 3\%(12{,}900 \times 1.03 = 13{,}287,\ \frac{13{,}287}{3} = \$4{,}429)$. What is the true rate of interest, $i^{(1)}$, you are being charged? [Hint: Calculate $i^{(4)}$ first.]

18. (SOA) A couple decides to save money for their child's first year of college. They wish to have at least \$10,000 is 5 years time. They deposit \$2,800 n months from today and another \$5,600 $2n$ months from today. If they can earn $i^{(12)} = 4.8\%$ on their deposits, what is maximum integral value n should be?

19. (SOA) You have two possible investments. The first one pays \$10,000 now and another \$10,815 two years from now. The second one pays \$20,800 one year from now. The present values of the two investments are equal. What annual effective interest rate is assumed?

20. (SOA) Dawn opens a bank account by making a deposit of \$1,000. Her account pays interest at $i^{(2)} = 6\%$. Shawn opens a bank account by making a deposit of \$1,000. His account pays interest at $i^{(12)} = 3\%$. For each account, interest is credited only at the end of an interest period. Calculate the number of months required for Dawn's account to be at least double the amount in Shawn's account.