ACTEX Academic Series

Theory of Interest and Life Contingencies with Pension Applications: A Problem-Solving Approach

Fourth Edition

4th

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and

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Theory of Interest and Life Contingencies With Pension Applications:

A Problem-Solving Approach

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PREFACE

It is impossible to escape the practical implications of compound interest in our modern society. The consumer is faced with a bewildering choice of bank accounts offering various rates of interest, and wishes to choose the one which will give the best return on her savings. A home-buyer is offered various mortgage plans by different companies, and wishes to choose the one most advantageous to him. An investor seeks to purchase a bond which pays coupons on a regular basis and is redeemable at some future date; again, there are a wide variety of choices available.

Comparing possibilities becomes even more difficult when the payments involved are dependent on the individual's survival. For example, an employee is offered a variety of different pension plans and must decide which one to choose. Also, most people purchase life insurance at some point in their lives, and a bewildering number of different plans are offered.

The informed consumer must be able to make an intelligent choice in situations like those described above. In addition, it is important that, whenever possible, she be able to make the appropriate calculations herself in such cases. For example, she should understand why a given series of mortgage payments will, in fact, pay off a certain loan over a certain period of time. She should also be able to decide which portion of a given payment is paying off the balance of the loan, and which portion is simply paying interest on the outstanding loan balance.

The first goal of this text is to give the reader enough information so that he can make an intelligent choice between options in a financial situation, and can verify that bank balances, loan payments, bond coupons, etc. are correct. Too few people in today's society understand how these calculations are carried out.

In addition, however, we are concerned that the student, besides being able to carry out these calculations, understands why they work. It is not enough to memorize a formula and learn how to apply it; you should understand why the formula is correct. We also wish to present the material in a proper mathematical setting, so the student will see how the theory of interest is interrelated with other branches of mathematics.

Let me explain why the phrase “Problem-Solving Approach” appears in the title of this text. We will prove a very small number of formulae and then concentrate our attention on showing how these formulae can be applied to a wide variety of problems. Skill will be needed to take the data presented in a particular problem and see how to rearrange it so the formulae can be used. This approach differs from many texts, where a large number of formulae are presented, and the student tries to memorize which problems can be solved by direct application of a particular formula. We wish to emphasize understanding, not rote memorization.

A working knowledge of elementary calculus is essential for a thorough understanding of all the material. However, a large portion of this book can be read by those without such a background by omitting the sections dependent on calculus. Other required background material such as geometric sequences, probability and expectation, is reviewed when it is required.

Each chapter in this text includes a large number of examples and exercises. The most efficient way for a student to learn the material is to work carefully through these exercises.

This text has appeared in three earlier editions, and before that a study note version was used for several years in a one semester Theory of Interest course at Memorial University of Newfoundland. I am deeply indebted to Brenda Crewe, Wanda Heath and P.P. Narayanaswami for help in preparing the original study notes.

Chuck Vinsonhaler was strongly supportive of the project and introduced me to the people at ACTEX Publications. I am very grateful to him for that.

Everyone at ACTEX was extremely helpful in the preparation of previous editions. I would like to especially thank Marilyn Baleshiski and Denise Rosencrant for their conscientious work in typesetting, and Marlene Lundbeck for designing the cover. Most importantly I would like to thank Dick London for his very helpful advice and assistance and for doing the technical content editing in all previous editions.

It’s amazing to me how a set of study notes, originally prepared almost 40 years ago for a Theory of Interest course at an isolated university off the east coast of Canada, is now going into a fourth edition as a textbook. For this new edition I’m delighted to welcome my co-author Kevin Shirley, and to sincerely thank him for his willingness to join this project. His expertise and enthusiasm are responsible for most of the many changes from previous editions, and we hope that this new revised and updated version will be helpful to students for many years to come.

Michael M. Parmenter

PREFACE to the Fourth Edition

In the twenty years since the third edition was published, significant changes have occurred in actuarial education to reflect the day-to-day work that actuaries are required to do. Many of these changes have their origin in the introduction of more complex products, especially those with embedded financial derivatives, the financial crises of 2007-2008, and the resulting regulatory responses. Today, actuaries regularly use more sophisticated software to model losses, perform simulations, and make economic projections. These changes have resulted in an increased reliance on technology and techniques that incorporate stochastic analysis, including simulation. These developments motivate some of the changes and additions made in the fourth edition. A discussion of the major modifications to the text follow.

One thing that has changed significantly in the last twenty years is that interest rates in all areas of financial life are much lower than they were before. As a consequence, most of the interest rates in the first few chapters of the text have been adjusted to better reflect current values. Except for these interest rate adjustments, and the addition of a number of new examples and exercises, Chapters 1 through 5 remain essentially unchanged.

In Chapter 6, a discussion of probability distributions has been added to the introduction to probability. Though the discussion is mostly in the context summarizing discrete distributions as tables, it lays the ground work for using random variables throughout the remainder of the text. In Section 6.4, the analysis in the determination of interest rates on loans has been expanded to include the language and understanding conveyed in the most recent SOA study note on the topic, *Determinates of Interest Rates*[1]. A discussion has been added to provide insight on how the interest rates used to discount the benefits in life insurances can be understood in relation to the insurance company's investments.

[1] Michael A. Bean (2017). *Determinates of Interest Rates.* Society of Actuaries.

Chapter 7 preserves the material from the third edition, but provides an introduction to the future lifetime random variable, Section 7.5, and the curtate future lifetime random variable, Section 7.7. In the third edition this material first appeared in Chapter 10. This earlier introduction of random variables is not intended to make the text more theoretical, but rather to allow the reader to make the connection between the concepts and the underlying variables as the concepts are introduced in Chapters 8 and 9. The chapter has also been broken up into more sections to allow for easier access to specific topics. In particular, the expectation of life, Section 7.8, has been fully revised.

The major change in Chapters 8 and 9 deals with computations. Previously, commutation functions were emphasized as the primary computation technique for calculating the values of net single premiums represented by the life annuity and life insurance symbols. In this edition, we preserve the commutation function method by moving it to the Appendix for those still using it and as a reference for the reader who may encounter it in the actuarial literature. However, in its place we provide an introduction to building actuarial tables using recursion formulas in a specified life table and using these tables to calculate values needed for solving specific problems. Most of the examples and exercises from the third edition are preserved and updated to use this technique. This calculation method is consistent with what many life and annuity actuaries use in practice when calculating values for net single premiums. Finally, a change in terminology from net single premium to expected present value is made to be consistent with the early introduction of random variables and the terminology as seen in actuarial texts covering more advanced material.

Chapter 10 includes much of the material from the third edition, but has been expanded to include sampling from the future lifetime distribution and simulation. Simulation is increasingly used by actuaries as they are called upon to perform principles-based calculations. In Chapter 11, a new section has been added, Section 11.4 Computational and Random Variable Considerations. This section extends the new material on calculation techniques to multi-life functions, as well as introduces the reader to the underlying future lifetime random variables. Chapter 12 on pension applications is largely unchanged.

A new chapter on general insurance has been added. Chapter 13 introduces the reader to basic loss models and premium and reserve determination in general insurance. Deductibles and policy limits are introduced as is their effect on the expected loss. The chapter includes sections on the positive payment model, Section 13.3, claim frequency models, Section 13.4, aggregate claims, Section 13.5, and gross premiums and reserves, Section 13.6. More than thirty exercises have been added to accompany this new material.

Most of the chapters include Extended Spreadsheet Exercises. These exercises provide the reader with a more in-depth experience solving problems in a spreadsheet environment. Microsoft EXCEL is used. They allow the reader to see how examples and problems from the text can be extended to a more realistic setting. These problems include constructing actuarial tables to a given life table, simulating life insurance benefits using the Gompertz model, extending examples beyond their pencil-and-paper analysis, and building a retirements analysis income worksheet.

Finally, I would like to thank Michael Parmenter who has included me as a co-author in this project and all the people at ACTEX who helped prepare this revision, especially Kim Neuffer who coordinated the project and Yijia Liu for his work in preparing and typesetting the document for publication. I would like to thank the reviewers and proofreaders, Michael Bean, Daniel Geiger, John Dinius and Michael Reilly. Without them there would be no fourth edition of what we hope is a reasonably comprehensive yet friendly and readable textbook for actuarial students.

Kevin L. Shirley

1

Interest: The Basic Theory

1.1 Accumulation Function

The simplest of all financial transactions is one in which an amount of money is invested for a period of time. The amount of money initially invested is called the principal and the amount it has grown to after the time period is called the accumulated value at that time. This is a situation which can easily be described by functional notation. If t is the length of time for which the principal has been invested, then the amount of money at that time will be denoted by $A(t)$. This is called the amount function. For the moment we will only consider values $t \geq 0$, and we will assume that t is measured in years. We remark that the initial value $A(0)$ is just the principal itself. In order to compare various possible amount functions, it is convenient mathematically to define the accumulation function from the amount function as $a(t) = \frac{A(t)}{A(0)}$. We note that $a(0) = 1$ and that $A(t)$ is just a constant multiple of $a(t)$, namely $A(t) = k \cdot a(t)$ where $k = A(0)$ is the principal.

What functions are possible accumulation functions? In theory, any function $a(t)$ with $a(0) = 1$ could represent the way in which money accumulates with the passage of time. Certainly, however, we would hope that $a(t)$ is increasing. Should $a(t)$ be continuous? That depends on the situation; if $a(t)$ represents the amount owing on a loan t years after it has been taken out, then $a(t)$ may be continuous if interest continues to accumulate for non-integer values of t. However, if $a(t)$ represents the amount of money in your bank account t years after the initial deposit (assuming no deposits or withdrawals in the meantime), then $a(t)$ will stay constant for periods of time, but will take a jump whenever interest is paid into the account. The graph of such an $a(t)$ will be a step function. We will normally assume in this text that $a(t)$ is continuous; it is easy to make allowances for other situations when they turn up.

In Figure 1.1 we have drawn graphs of three different types of accumulation functions which occur in practice:

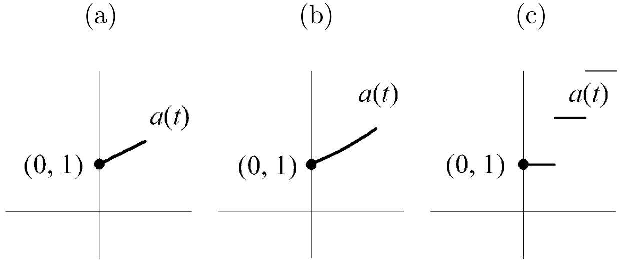

Figure 1.1

Graph (a) represents the case where the amount of interest earned is constant over each year. On the other hand, in cases like (b), the amount of interest earned is increasing as the years go on. This makes more sense in most situations, since we would hope that as the principal gets larger, the interest earned also increases; in other words, we would like to be in a situation where "interest earns interest." There are many different accumulation functions which look roughly like the graph in (b), but the exponential curve represents compound interest and is the one which will be of greatest interest to us.

We remarked earlier that a situation like (c) can arise whenever interest is paid out at fixed periods of time, but no interest is paid if money is withdrawn between these time periods. If the amount of interest paid is constant per time period, then the "steps" will all be of the same height. However, if the amount of interest increases as the accumulated value increases, then we would expect the steps to get larger and larger as time goes on. We have used the term interest several times now, so perhaps it is time to define it!

Interest= Accumulated Value − Principal

This definition is not very helpful in practical situations, since we are generally interested in comparing different financial situations to determine which is most profitable. What we require is a standardized measure for interest, and we do this by defining the effective rate of interest i (per period) to be the interest earned on a principal of amount 1 over a period (often of one year). That is,

$$i = a(1) - a(0) = a(1) - 1. \tag{1.1}$$

We can easily calculate i using the amount function $A(t)$ instead of $a(t)$, if we recall that $A(t) = k \cdot a(t)$. Thus

$$i = a(1) - 1 = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)}. \tag{1.2}$$

Verbally, the effective rate of interest per period is the amount of interest earned in one period divided by the principal at the beginning of the period.

More generally, we define the effective rate of interest in the n^{th} period by

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)} \tag{1.3}$$

Note that i_1, calculated by (1.3), is the same as i defined by either (1.1) or (1.2).

Example 1.1. Consider the function $a(t) = t^2 + t + 1$.

(a) Verify that $a(0) = 1$.

(b) Show that $a(t)$ is increasing for all $t \geq 0$.

(c) Is $a(t)$ continuous?

(d) Find the effective rate of interest i for $a(t)$.

(e) Find i_n.

Solution. (a) $a(0) = (0)^2 + (0) + 1 = 1$.

(b) Note that $a'(t) = 2t + 1 > 0$ for all $t \geq 0$, so $a(t)$ is increasing.

(c) The easiest way to solve this is to observe that the graph of $a(t)$ is a parabola, and hence $a(t)$ is continuous (or recall from calculus that all polynomial functions are continuous).

(d) $i = a(1) - 1 = 3 - 1 = 2$. Often this is expressed as 200%.

(e) $i_n = \frac{a(n)-a(n-1)}{a(n-1)} = \frac{n^2+n+1-[(n-1)^2+(n-1)+1]}{(n-1)^2+(n-1)+1} = \frac{2n}{n^2-n+1}$.

□

1.2 Simple Interest

There are two special cases of the accumulation function $a(t)$ that we will examine closely. The first of these, simple interest, is used occasionally, primarily between integer interest periods, but will be discussed mainly for historical purposes and because it is easy to describe. The second of these, compound interest, is by far the most important accumulation function and will be discussed in the next section. Keep in mind that in both of these cases $a(t)$ is continuous, and also that there are some practical settings where modifications must be made.

Simple interest is the case where the graph of $a(t)$ is a straight line. Since $a(0) = 1$, the equation must therefore be of the general form $a(t) = 1 + bt$ for some b. However, the effective rate of interest i is given by $i = a(1) - 1 = b$, so the formula is

$$a(t) = 1 + it, t \geq 0. \tag{1.4}$$

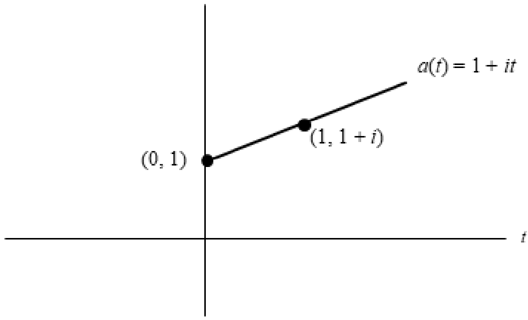

Figure 1.2

Remarks

1. This is case (a) graphed in Figure 1.1 In this situation, the amount of interest earned each year is constant. In other words, only the original principal earns interest from year to year, and interest accumulated in any given year does not earn interest in future years.

2. The formula $a(t) = 1 + it$ applies to the case where the principal is $A(0) = a(0) = 1$. More generally, if the principal at time 0 is equal to k, the amount at time t will be $A(t) = k(1 + it)$.

3. We noted above that the "i" in $a(t) = 1 + it$ is also the effective rate of interest for this function. Note however that

$$i_n = \frac{1 + in - [1 + i(n-1)]}{1 + i(n-1)} = \frac{i}{1 + i(n-1)}. \tag{1.5}$$

   Observe that i_n is not constant. In fact, i_n decreases as n gets larger, a fact which should not surprise us. If the monetary amount of interest stays constant as the accumulated value increases, then clearly the effective rate of interest is going down.

4. Clearly $a(t) = 1 + it$ is a formula which works equally well for all values of t, integral or otherwise. However, problems can develop in practice, as illustrated by the following example.

Example 1.2. Assume Jack borrows 1000 from the bank on January 1, 2021 at a rate of 5% simple interest per year. How much does he owe on January 17, 2021?

Solution. The general formula for the amount owed at time t in general is $A(t) = 1000(1 + .05t)$, but the problem is to decide what value of t should be substituted into this formula. An obvious approach is to take the number of days which have passed since the loan was taken out and divide by the number of days in the year, but should we count the number of days as 16 or 17? Getting really picky, should we worry about the time of day when the loan was taken out, or the time of day when we wish to find the value of the loan? Obviously, any value of t is only a convenient approximation; the important thing is to have a consistent rule to be used in practice.
The commonest method is

$$t = \frac{number\ of\ days}{365}. \tag{1.6}$$

When counting the number of days it is usual to count the last day, but not the first. In our case this would lead to $t = \frac{16}{365}$ so Jack owes $1000\left[1 + (.05)(\frac{16}{365})\right] = 1002.19$. □

1.3 Compound Interest

The most important special case of the accumulation function $a(t)$ is the case of compound interest. Intuitively speaking, this is the situation where money earns interest at a fixed effective rate; in this setting, the interest earned in one year earns interest itself in future years.

If i is the effective rate of interest, we know that $a(1) = 1 + i$, so 1 becomes $1 + i$ after the first year. What happens in the second year? Consider the $1 + i$ as consisting of two parts, the initial principal 1 and the interest i earned in the first year. The principal 1 will earn interest in the second year and will accumulate to $1 + i$. The interest i will also earn interest in the second year and will grow to $i(1 + i)$. Hence the total amount after two years is $1 + i + i(1 + i) = (1 + i)^2$. By continuing this reasoning, we see that the formula for $a(t)$ is

$$a(t) = (1 + i)^t, t \geq 0. \tag{1.7}$$

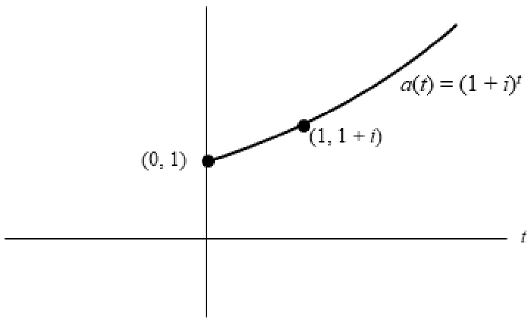

Figure 1.3

Remark

1. This is an example of the type of function shown in part (b) of the graph in Figure 1.1
2. The formula $a(t) = (1 + i)^t$ applies to the case where the principal is $A(0) = a(0) = 1$. More generally, if the principal at time 0 is equal to k, the amount at time t will be $A(t) = k(1 + i)^t$.
3. Observe that the "i" in $(1 + i)^t$ is the effective rate of interest. More generally,

$$i_n = \frac{(1 + i)^n - (1 + i)^{n-1}}{(1 + i)^{n-1}} = 1 + i - 1 = i. \tag{1.8}$$

   Hence, in this case i_n is the same for all positive integers n. We shouldn't be surprised, since this fits with our idea that, in compound interest, the effective rate of interest is constant.
4. Mathematically, any value of t, whether integral or not, can be substituted into $a(t) = (1 + i)^t$. This is an easier task for us today than it was many years ago; we just have to press the appropriate buttons on our calculators! Again, there are problems determining what value of t should be used, but we can deal with them as we did in the last section.

In practical situations, however, a very different solution is sometimes used in the case of compound interest. To find the amount of a loan (for example) when t is a fraction, first find the amounts for the integral values of t immediately before and immediately after the fractional value in question. Then use linear interpolation between the two computed amounts to calculate the required answer.

This is equivalent to saying that compound interest is used for integral values of t, and simple interest is used between integral values. In Figure 1.4, the solid line represents $a(t) = (1+i)^t$, whereas the dotted lines indicate the graph of $a(t)$ if linear interpolation is used.

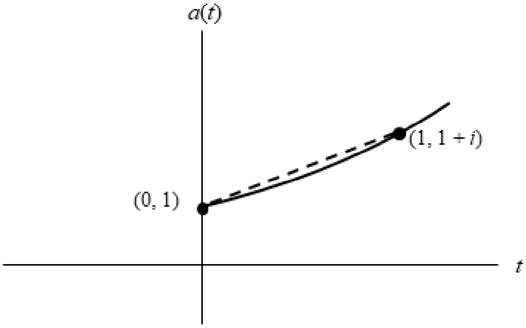

Figure 1.4

As we will see later, this common procedure benefits the lender in a financial transaction, and (consequently) is detrimental to the borrower if she has to repay the loan at a duration between integral values.

Example 1.3. Jack borrows 1000 at 5% compound interest.

(a) How much does he owe after 2 years?

(b) How much does he owe after 57 days, assuming compound interest between integral durations?

(c) How much does he owe after 1 year and 57 days, under the same assumption as in part (b)?

(d) How much does he owe after 1 year and 57 days, assuming linear interpolation between integral durations?

(e) In how many years will his principal (i.e. debt) have accumulated to 2000?

Solution. (a) $1000(1.05)^2 = 1102.50$.

(b) The most suitable value for t is $\frac{57}{365}$, and the accumulated value is $1000(1.05)^{\frac{57}{365}} = 1007.65$.

(c) $1000(1.05)^{1\frac{57}{365}} = 1058.03$.

(d) We must interpolate between $A(1) = (1000)(1.05) = 1050.00$ and $A(2) = 1000(1.05)^2 = 1102.50$. The difference between these values is $A(2) - A(1) = 52.50$. The portion of this difference which will accumulate in 57 days, assuming simple interest, is $\left(\frac{57}{365}\right)(52.50) = 8.20$. Thus the accumulated value after 1 year and 57 days is $1050.00 + 8.20 = 1058.20$. Observe that the borrower owes more money in this case than he does in part (c).

(e) We seek t such that $1000(1.05)^t = 2000$, or that $(1.05)^t = 2$. Using logs we obtain

$$t = \frac{\log\ 2}{\log 1.05} = 14.2067 \text{ years.}$$

□

Example 1.4. Irving and Yuri open up new bank accounts on January 1, 2015. Irving begins with 100 in his account and Yuri begins with 150 in his. Both accounts earn money at the same effective annual rate of compound interest i. The amount of interest earned in Irving's account during the 21st year equals the amount of interest earned in Yuri's account during the 11th year. Find i.

Solution. The amount of interest earned by Irving in the 21st year is $100(1+i)^{21} - 100(1+i)^{20} = 100(1+i)^{20}i$.
The amount of interest earned by Yuri in the 11th year is $150(1+i)^{11} - 150(1+i)^{10} = 150(1+i)^{10}i$.
So we have $100(1+i)^{20}i = 150(1+i)^{10}i$, giving $(1+i)^{10} = 1.5$.
Therefore, $i = 1.5^{\frac{1}{10}} - 1 = .04138$. □

To close this section, we will compare simple interest and compound interest to see which gives the better return. In Figure 1.5, graphs for both simple interest and compound interest are drawn on the same set of axes.

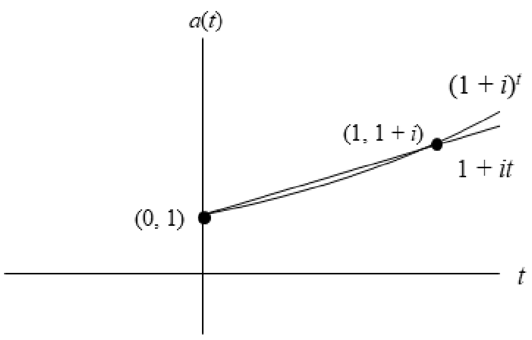

Figure 1.5

We know that the exponential function $(1+i)^t$ is always concave up (because the second derivative is $(1+i)^t[\ln(1+i)]^2$, which is greater than zero), whereas $1+it$ is a straight line. These facts tell us that the only points of intersection of these graphs are the obvious ones, namely (0,1) and $(1, 1+i)$. They also give us the two important relationships

$$(1+i)^t < 1+it, \text{ for } 0 < t < 1, \tag{1.9}$$

and

$$(1+i)^t > 1+it, \text{ for } t > 1. \tag{1.10}$$

Hence we conclude that compound interest yields a higher return than simple interest if $t > 1$, whereas simple interest yields more if $0 < t < 1$. The first of these statements does not surprise us, since for $t > 1$, we have interest as well as principal earning interest in the $(1+i)^t$ case. The second statement reminds us that, for periods of less than a year, simple interest is more beneficial to the lender or investor than compound interest, a fact which was illustrated in Example 1.3.

1.4 Present Value and Discount

In Section 1.1 we defined accumulated value at time t as the amount that the principal accumulates to over t years. We now define the present value t years in the past as the amount of money that will accumulate to the principal over t years. In other words, this is the reverse procedure of that which we have been discussing up to now.

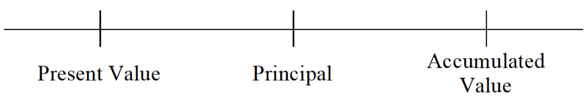

Figure 1.6

For example, 1 accumulates to $1+i$ over a single year. How much money is needed, at the present time, to accumulate to 1 over one year? We will denote this amount by v, and, recalling that v accumulates to $v(1+i)$, we have $v(1+i) = 1$. Therefore

$$v = \frac{1}{1+i}. \tag{1.11}$$

These two accumulations are shown in Figure 1.7.

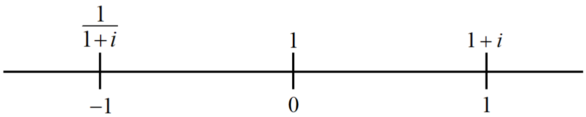

Figure 1.7

From now on, unless explicitly stated otherwise, we will assume that we are in a compound interest situation, where $a(t) = (1+i)^t$. In this case, the present value of 1, t years in the past, will be $v^t = \frac{1}{(l+i)^t}$. We summarize this on the time diagram shown in Figure 1.8.

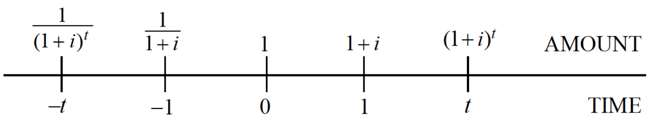

Figure 1.8

Observe that, since $v^t = (1+i)^{-t} = a(-t)$, the function $a(t) = (1+i)^t$ expresses all these values, for both positive and negative values of t. Hence $(1+i)^t$ gives the value of one unit (at time 0) at any time t, past or future. The graph is shown in Figure 1.9.

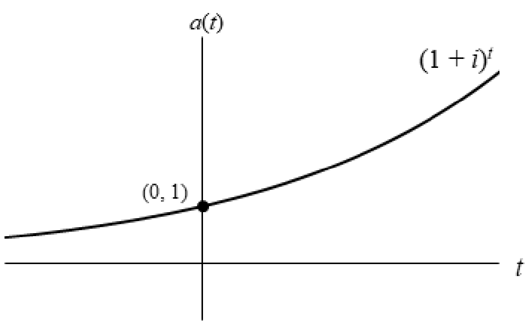

Figure 1.9

Example 1.5. The Kelly family buys a new house for 93500 on May 1, 2025. How much was this house worth on May 1, 2021, if real estate prices have risen at a compound rate of 4% per year during that period?

Solution. We seek the present value, at time $t = -4$, of 93500 at time 0. This is $93500(\frac{1}{1.04})^4 = 79924.19$. □

What happens to the calculation of present values if simple interest is assumed instead of compound interest? The accumulation function is now $a(t) = 1 + it$. Hence, the present value of one unit t years in the past is given by x, where $x(1 + it) = 1$. Thus the present value is

$$x = \frac{1}{1 + it}. \tag{1.12}$$

The time diagram for this case is shown in Figure 1.10.

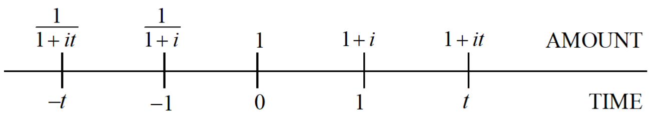

Figure 1.10

In Exercise 15, you are asked to sketch the graph of this situation. Unlike the compound interest case, this graph changes dramatically as it passes through the point (0,1).

We now turn our attention to the concept of discount. For the moment we will not assume compound interest, since any accumulation function will be satisfactory.

Imagine that 100 is invested, and that one year later it has accumulated to 104.20. We have been viewing the 100 as the "starting figure", and have imagined that interest of 4.20 is added to it at the end of the year. However, we could also view 104.20 as the basic figure, and imagine that 4.20 is deducted from that value at the start of the year. From the latter point of view, the 4.20 is considered an amount of discount.

Students sometimes get confused about the difference between interest and discount, but the important thing to remember is that the only difference is in the point of view, not in the underlying financial transaction. In both situations we have 100 accumulating to 104.20, and nothing can change that.

Since discount focuses on the total at the end of the year, it is natural to define the effective rate of discount, d, as

$$d = \frac{a(1) - a(0)}{a(1)} = \frac{a(1) - 1}{a(1)}. \tag{1.13}$$

In other words, standardization is achieved by dividing by $a(1)$ instead of $a(0)$, as was done in (1.2) to define the effective rate of interest i.

More generally, the effective rate of discount in the n^{th} year is given by

$$d_n = \frac{a(n) - a(n-1)}{a(n)}. \tag{1.14}$$

(Compare this with the definition of i_n, given by (1.3).)

Now we will derive some basic identities relating d to i. One identity follows immediately from the definition of d, namely,

$$d = \frac{a(1) - 1}{a(1)} = \frac{(1+i) - 1}{1+i} = \frac{i}{1+i}. \tag{1.15}$$

Since $1 + i > 1$, this tells us that $d < i$.

Immediately from the above we obtain

$$1 - d = 1 - \frac{i}{1+i} = \frac{1}{1+i} = v. \tag{1.16}$$

Actually, this identity is exactly what we would expect from the definition of d. The fact that $1 - d$ accumulates to 1 over one year is the exact analogy of 1 accumulating to $1 + i$ over the same period.

Solving either of the above identities for i, we obtain

$$i = \frac{d}{1-d}. \tag{1.17}$$

The reader will be asked to derive other identities in the exercises and to give verbal arguments in support of them. We note that all identities derived so far hold for any accumulation function. For the rest of this section, it will be assumed that $a(t) = (1+i)^t$.

In Section 1.3 we learned that to find the accumulated value t years in the future we multiply by $(1+i)^t$, whereas to find the present value t years in the past we multiply by the discount factor $\frac{1}{(1+i)^t}$. However, identity (1.17) tells us that $1 - d = \frac{1}{1+i}$. Hence, if d is involved, the rules for present and accumulated value are reversed: present value is obtained by multiplying by the discount factor $(1-d)^t$, and accumulated value by multiplying by $\frac{1}{(1-d)^t}$.

Example 1.6. 1000 is to be accumulated by January 1, 2025, at a compound rate of discount of 4.5% per year.

(a) Find the present value on January 1, 2022.

(b) Find the value of i corresponding to d.

Solution. (a) $1000(1-.045)^3 = 870.98$.

(b) $i = \frac{d}{1-d} = \frac{.045}{.955} = .0471$.

□

Example 1.7. Jane deposits 1000 in a bank account on August 1, 2023. If the rate of compound interest is 3.9% per year, find the value of this deposit on August 1, 2021.

Solution. Some students think that the answer to this question should be 0, because the money has not been deposited yet! However, in a mathematical sense, we know that money has value at all times, past or future, so the correct answer is $1000(\frac{1}{1.039})^2 = 926.34$. □

1.5 Nominal Rate of Interest

We will assume $a(t) = (1+i)^t$ throughout this section and, unless stated otherwise, in all remaining sections of the book.

Example 1.8. A man borrows 1000 at an effective rate of interest of 0.5% per month. How much does he owe after 3 years?

Solution. What we want is the amount of the debt after three years. Since the effective interest rate is given per month, three years is 36 interest periods. Thus the answer is $1000(1.005)^{36} = 1196.68$. □

The point of the above example is to illustrate that effective rates of interest need not be given per year, but can be defined with respect to any period of time. To apply the formulae developed to this point, we must be sure that t is the number of effective interest periods in any particular problem.

In many real-life situations, the effective interest period is not a year, but rather some shorter period. Perhaps the lender tries to keep this fact hidden, as it might be to his benefit to do so! For example, suppose you want to take out a mortgage on a house and you discover a rate of 4% per year. When you dig a little, however, what you find out is that this rate is “convertible semiannually”, which means that it is really 2% effective per half-year. Is that the same thing? Not at all. Consider what happens to an investment of 1. After half a year it has accumulated to 1.02. After one year (two interest periods) it has become $(1.02)^2 = 1.0404$. So, over a one-year period, the amount of interest gained is .0404, which means the effective rate of interest per year is actually 4.04%. Although it may not be clear from the advertising, many mortgage loans are convertible semiannually, so the effective rate of interest is higher than the rate quoted.

As another example, consider a well-known credit card which charges 5.4% per year convertible monthly. This means that the actual rate of interest is $\frac{.054}{12} = .0045$ effective per month. Over the course of a year, 1 will accumulate to $(1.0045)^{12} = 1.0554$, so the effective rate of interest per year is actually 5.54%.

The 5.4% in the last example is called a nominal rate of interest, which means that it is convertible over a period other than one year. In general, we use the notation $i^{(m)}$ to denote a nominal rate of interest convertible m times per year, which implies an effective rate of interest of $\frac{i^{(m)}}{m}$ per m^{th} of a year. If i is the effective rate of interest per year, it follows that

$$1 + i = \left[1 + \frac{i^{(m)}}{m}\right]^m. \tag{1.18}$$

Example 1.9. Find the accumulated value of 1000 after three years at a nominal rate of interest of 6% per year convertible monthly.

Solution. This is really 0.5% effective per month, so the answer is the same as Example 1.8, namely $1000(1.005)^{36} = 1196.68$. □

Remark

An alternative method of solving Example 1.9 is to find i, the effective rate of interest per year, and then proceed as in Section 1.3. We would have $i = (1 + \frac{i^{(m)}}{m})^m - 1 = (1 + .005)^{12} - 1 = .06168$, and the answer would be $1000(1.06168)^3 = 1196.69$.

Notice the difference of .01 in the two answers. This is because not enough decimal places were kept in the value of i, and some error crept in. Of course, if you use the memory in your calculator it is unlikely that this type of error will occur. Nevertheless, the first solution is still preferable; time spent on unnecessary calculations can be significant in examination situations.

It will be extremely important in later sections of the text to be able to convert from one nominal rate of interest to another whose convertible frequency is different. Here is an example of this.

Example 1.10. If $i^{(6)} = .048$, find the equivalent nominal rate of interest convertible semiannually.

Solution. We have $(1 + \frac{i^{(2)}}{2})^2 = (1 + \frac{.048}{6})^6$, so $i^{(2)} = 2[(1.008)^3 - 1] = .04839$. □

In the same way that we defined a nominal rate of interest, we could also define a nominal rate of discount, $d^{(m)}$, as meaning an effective rate of discount of $\frac{d^{(m)}}{m}$ per m^{th} of a year. Analogous to identity (1.18), it is easy to see that

$$1 - d = \left[1 - \frac{d^{(m)}}{m}\right]^m. \tag{1.19}$$

Since $1 - d = \frac{1}{1+i}$, we conclude that

$$\left[1 + \frac{i^{(m)}}{m}\right]^m = 1 + i = (1-d)^{-1} = \left[1 - \frac{d^{(n)}}{n}\right]^{-n} \tag{1.20}$$

for all positive integers m and n.

Example 1.11. Find the nominal rate of discount convertible semi-annually which is equivalent to a nominal rate of interest of 4.2% per year convertible monthly.

Solution. $\left[1 - \frac{d^{(2)}}{2}\right]^{-2} = \left[1 + \frac{i^{(12)}}{12}\right]^{12}$, so

$$1 - \frac{d^{(2)}}{2} = (1.0035)^{-6} = .979255,$$

from which we find $d^{(2)} = 2(1 - .979255) = .04149$. □

1.6 Force of Interest

We note before starting this section that it is somewhat theoretical, and is independent of the rest of the text. Anyone wishing to proceed directly to more practical problems can safely omit this material. In particular, more background knowledge is required for a full understanding here than is required for any other section; students with only a sketchy knowledge of calculus might omit this on first reading.

Assume that the effective annual rate of interest is $i = .04$, and that we want to find nominal rates $i^{(m)}$ equivalent to i. The formula $i^{(m)} = m\left[(1+i)^{1/m} - 1\right]$, which comes from identity (1.18), is used to calculate these values which are shown in Table 1.1.

m	1	2	5	10	50
$i^{(m)}$	.04	.03961	.03937	.03930	.03924

Table 1.1

We observe that $i^{(m)}$ decreases as m gets larger, a fact which we will be able to prove later in this section. We also observe that the values of $i^{(m)}$ are decreasing very slowly as we go further and further along; in the language of calculus, $i^{(m)}$ seems to be approaching a limit. This is, in fact, what is happening, and we can use L'Hopital's rule to see what the limit is. There is no need to assume $i = .04$ in our derivation, so we proceed with arbitrary i.

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} m[(1+i)^{\frac{1}{m}} - 1] = \lim_{m\to\infty} \frac{(1+i)^{\frac{1}{m}} - 1}{\frac{1}{m}}. \tag{1.21}$$

Since (1.21) is of the form $\frac{0}{0}$, we take derivatives top and bottom, cancel, and obtain

$$\lim_{m\to\infty} i^{(m)} = \lim_{m\to\infty} [(1+i)^{1/m} \cdot \ln(1+i)] = \ln(1+i) \tag{1.22}$$

since $\lim_{m\to\infty} (1+i)^{1/m} = 1$. This limit is called the force of interest and is denoted by δ, so we have

$$\delta = \ln(1+i). \tag{1.23}$$

In our example, $\delta = \ln(1.04) = .03922$. The reader should compare this with the entries in Table 1.1.

Intuitively, δ represents a nominal rate of interest which is convertible continuously, a notion of more theoretical than practical importance. However, δ can be a very good approximation for $i^{(m)}$ when m is large (for example, a nominal rate convertible daily), and has the advantage of being very easy to calculate.

We note that identity (1.23) can be rewritten as

$$e^{\delta} = 1 + i. \tag{1.24}$$

The usefulness of this form is shown in the next example. Again we stress the importance of being able to convert a rate of interest with a given conversion frequency to an equivalent rate with a different conversion frequency.

Example 1.12. A loan of 3000 is taken out on June 23, 2017. If the force of interest is 4%, find each of the following:

(a) The value of the loan on June 23, 2022.

(b) The value of i.

(c) The value of $i^{(12)}$.

Solution. (a) The value 5 years later is $3000(1+i)^5$. Using $e^\delta = 1+i$, we obtain $3000(e^{.04})^5 = 3000e^{.2} = 3664.21$.

(b) $i = e^{.04} - 1 = .04081$.

(c) $(1+\frac{i^{(12)}}{12})^{12} = 1+i = e^{.04}$,
so we have the result $i^{(12)} = 12(e^{.04/12} - 1) = .04007$.

□

Remark

Note that if we tried to solve part (a) by first obtaining $i = .04081$ (as in part (b)), and then calculating $3000(1.04081)^5$, we would get 3664.19, an answer differing from our first answer by .02. There is nothing wrong with this second method, except that not enough decimal places were carried in the value of i to guarantee an accurate answer. Let us repeat an earlier admonition: it is always wise to do as few calculations as necessary. Observe that $\frac{d}{dt}[(1+i)^t] = (1+i)^t \cdot \ln(1+i)$. Hence we see that

$$\delta = \ln(1+i) = \frac{\frac{d}{dt}[(1+i)^t]}{(1+i)^t} = \frac{\frac{d}{dt}[a(t)]}{a(t)}. \tag{1.25}$$

Let us see why this fact happens to be true. Recall from the definition of the derivative that $\frac{d}{dt}[a(t)] = \lim\limits_{h\to 0} \frac{a(t+h)-a(t)}{h}$, so

$$\frac{\frac{d}{dt}[a(t)]}{a(t)} = \lim_{h\to 0} \frac{a(t+h)-a(t)}{h \cdot a(t)} = \lim_{h\to 0} \frac{\frac{a(t+h)-a(t)}{a(t)}}{h}. \tag{1.26}$$

The term $\frac{a(t+h)-a(t)}{a(t)}$ in (1.26) is just the effective rate of interest over a very small time period h, so $\frac{\frac{a(t+h)-a(t)}{a(t)}}{h}$ is the nominal annual rate corresponding to that effective rate, which agrees with our earlier definition of δ.

The above analysis does more than that, however. It also indicates how the force of interest should be defined for arbitrary accumulation functions.

First, let us observe that $\delta = \ln(1+i)$ is independent of t. However, this is a special property of compound interest, corresponding to a constant i_n. For arbitrary accumulation functions, we define the force of interest at time t, δ_t, by

$$\delta_t = \frac{\frac{d}{dt}[a(t)]}{a(t)}. \tag{1.27}$$

since we would normally expect δ_t to depend on t.

For certain functions, it is more convenient to use the equivalent definition

$$\delta_t = \frac{d}{dt}[\ln(a(t))]. \tag{1.28}$$

We also remark that, since $A(t) = k \cdot a(t)$, it follows that

$$\delta_t = \frac{\frac{d}{dt}[A(t)]}{A(t)} = \frac{d}{dt}[\ln(A(t))]. \tag{1.29}$$

Example 1.13. Find δ_t in the case of simple interest.

Solution.

$$\delta_t = \frac{D(1+it)}{1+it} = \frac{i}{1+it}.$$

□

We now have a method for finding the force of interest, δ_t, given any accumulation function $a(t)$. What if we are given δ_t instead, and wish to derive $a(t)$ from it?

To start with, let us write our definition of δ_t from (1.29) using a different variable, namely $\delta_r = \frac{d}{dr}[\ln(a(r))]$. Integrating both sides of this equation from 0 to t, we obtain

$$\begin{aligned} \int_0^t \delta_r dr &= \int_0^t \frac{d}{dr}[\ln(a(r))]dr \\ &= \ln(a(r))|_0^t \\ &= \ln(a(t)) - \ln(a(0)) \\ &= \ln(a(t)) \end{aligned} \tag{1.30}$$

since $a(0) = 1$ and $\ln 1 = 0$. Then taking the antilog we have

$$a(t) = e^{\int_0^t \delta_r dr}. \tag{1.31}$$

Example 1.14. Prove that if δ is a constant (i.e., independent of r), then $a(t) = (1+i)^t$ for some i.

Solution. If $\delta_r = c$, the right hand side of (1.31) is $e^{\int_0^t c dr} = e^{ct} = (e^c)^t$. Hence the result is proved with $i = e^c - 1$. □

Example 1.15. Chandra makes deposits of 100 at time 0 and X at time 4. His account has a force of interest of $\delta_t = \frac{t^2}{400}$. The amount of interest earned from time 4 to time 7 is $2X$. Find X.

Solution. The 100 deposit accumulates to $100e^{\int_0^4 \frac{t^2}{400}dt} = 100e^{\frac{64}{1200}} = 105.48$ at time 4, and to $100e^{\int_0^7 \frac{t^2}{400}dt} = 133.09$ at time 7. Hence the interest earned by the 100 from time 4 to time 7 is $133.09 - 105.48 = 27.61$. The deposit of X accumulates to $Xe^{\int_4^7 \frac{t^2}{400}dt} = 1.2617504X$ at time 7, so the interest earned is $0.2617504X$.

Thus $2X = 0.2617504X + 27.61$, and solving for X we obtain $X = 15.88$. $\square$

Example 1.16. Prove that $\int_0^n A(t)\delta_t dt = A(n) - A(0)$ for any amount function $A(t)$.

Solution. The left hand side is $\int_0^n A(t)\delta_t dt = \int_0^n A(t)\left[\frac{\frac{d}{dt}[A(t)]}{A(t)}\right] dt = \int_0^n \frac{d}{dt}\left[A\left(t\right)\right] dt = A(t)|_0^n = A(n) - A(0)$ as required. $\square$

The identity in the above example has an interesting verbal interpretation. The term $\delta_t dt$ represents the effective rate of interest at time t for the infinitesimal "period of time" dt. Hence $A(t)\delta_t dt$ represents the amount of interest earned in this period, and $\int_0^n A(t)\delta_t dt$ represents the total amount of interest earned over the entire period, a number which is clearly equal to $A(n) - A(0)$.

The next example introduces the notion of payments being made continuously into an account. Finding the accumulated value of such a set of continuous payments requires integration.

Example 1.17. George makes deposits into his account at a continuous rate of $20k + tk$ where $0 \leq t \leq 8$. The account has a force of interest of $\delta_t = \frac{1}{t+20}$. After 8 years George has 10000 in the account. Find k.

Solution. At time $r, 0 \leq r \leq 8$, an infinitesimal deposit of $(20k+rk)dr$ is made, and this deposit accumulates to $(20k+rk)dr\ e^{\int_r^8 \frac{1}{t+20}dt}$ by the end of year 8. Note that

$$\begin{aligned}(20k + rk)(e^{\int_r^8 \frac{1}{t+20}dt})dr &= (20k + rk)e^{\ln 28 - \ln(r+20)}\ dr \\ &= k(20 + r)e^{\ln \frac{28}{r+20}}\ dr \\ &= k(20 + r)(\frac{28}{r + 20})\ dr = 28kdr.\end{aligned}$$

The accumulated value at $t = 8$ of all such deposits is $\int_0^8 28kdr = (28k)8 = 224k$. So $224k = 10000$, giving $k = 44.643$. $\square$

We now return to the compound interest case where we have $a(t) = (1+i)^t$. It is interesting to write some of the formulae already developed as power series expansions. For example $\delta = \ln(1+i)$ becomes

$$\delta = i - \frac{i^2}{2} + \frac{i^3}{3} - \frac{i^4}{4} + \cdots . \tag{1.32}$$

Convergence is a concern here, but as long as $|i| < 1$, which is usually the case, the above series does converge.

Another important formula was $i = e^\delta - 1$, which becomes

$$i = \delta + \frac{\delta^2}{2!} + \frac{\delta^3}{3!} + \cdots . \tag{1.33}$$

Since all terms on the right hand side are positive, this allows us to conclude immediately that $i > \delta$. We note in passing that this series converges for all δ.

Next let us expand the expression $i = \frac{d}{1-d} = d(1-d)^{-1}$, which becomes

$$i = d(1 + d + d^2 + d^3 + \cdots) = d + d^2 + d^3 + \cdots . \tag{1.34}$$

Again this shows us very clearly that $i > d$. We also note that we must have $|d| < 1$ for this series to converge. In fact, trying to put $d = 2$ yields an amusing result: the left hand side is $i = \frac{2}{1-2} = -2$, whereas the right hand side becomes $2 + 2^2 + 2^3 + \cdots$, all of which are positive terms. Thus we have "proven" that -2 is a positive number!

Next let us expand $i^{(m)}$ as a function of i. From (1.19) we have $i^{(m)} = m[(1+i)^{1/m} - 1]$, so

$$\begin{aligned} i^{(m)} &= m\left[1 + \frac{1}{m}i + \frac{\frac{1}{m}(\frac{1}{m}-1)}{2!}i^2 + \frac{(\frac{1}{m})(\frac{1}{m}-1)(\frac{1}{m}-2)}{3!}i^3 + \cdots - 1\right] \\ &= i + \left[\frac{\frac{1}{m}-1}{2!}\right]i^2 + \frac{(\frac{1}{m}-1)(\frac{1}{m}-2)}{3!}i^3. \end{aligned} \tag{1.35}$$

Again, this converges for $|i| < 1$.

Why are we interested in power series expansions? Well, we have already seen that they sometimes allow us to easily conclude facts like $i > \delta$ (although they certainly are not needed for that). They also give us a quick means of calculating some of these functions, since often only the first few terms of the series are necessary for a high degree of accuracy. If you ask your calculator to do this work for you instead, it will oblige, but the program used for the calculation will often be a variation of one of those described above.

As a final example, let us expand $d^{(m)}$ in terms of δ. We have

$$\left[1 - \frac{d^{(m)}}{m}\right]^m = (1+i)^{-1} = e^\delta \tag{1.36}$$

so

$$\begin{aligned}
d^{(m)} &= m[1 - e^{-\delta/m}] \\
&= m\left[1 - (1 + (-\frac{\delta}{m}) + \frac{\left(-\frac{\delta}{m}\right)^2}{2!} + \frac{\left(-\frac{\delta}{m}\right)^3}{3!} + \cdots)\right] \\
&= m\left[\frac{\delta}{m} - \frac{\delta^2}{2!m^2} + \frac{\delta^3}{3!m^3} - \cdots\right] \\
&= \delta - \frac{\delta^2}{2!m} + \frac{\delta^3}{3!m^2} - \cdots.
\end{aligned}$$

From this we easily see that $\lim_{m\to\infty} d^{(m)} = \delta$. In other words, there is no need to define a force of discount, because it will turn out to be the same as the force of interest already defined.

EXERCISES

1.1 Accumulation Function; 1.2 Simple Interest; 1.3 Compound Interest

1-1. Alphonse has 14,000 in an account on January 1, 2020.

(a) Assuming simple interest at 3% per year, find the accumulated value on January 1, 2026.

(b) Assuming compound interest at 3% per year, find the accumulated value on January 1, 2026.

(c) Assuming simple interest at 3% per year, find the accumulated value on March 8, 2020.

(d) Assuming compound interest at 3% per year, but linear interpolation between integral durations, find the accumulated value on February 17, 2022.

1-2. Mary has 14,000 in an account on January 1, 2020.

(a) Assuming compound interest at 3.8% per year, find the accumulated value on January 1, 2025.

(b) Assuming simple interest at 3.8% per year, find the accumulated value on April 7, 2025.

(c) Assuming compound interest at 3.8% per year, but linear interpolation between integral durations, find the accumulated value on April 7, 2025.

1-3. For the $a(t)$ function given in Example 1.1, prove that $i_{n+1} < i_n$ for all positive integers n.

1-4. Consider the function $a(t) = \sqrt{1 + (i^2 + 2i)t^2}, i > 0, t \geq 0$.

(a) Show that $a(0) = 1$ and $a(1) = 1 + i$.

(b) Show that $a(t)$ is increasing and continuous for $t \geq 0$.

(c) Show that $a(t) < 1 + it$ for $0 < t < 1$, but $a(t) > 1 + it$ for $t > 1$.

(d) Show that $a(t) < (1 + i)^t$ if t is sufficiently large.

1-5. Let $a(t)$ be a function such that $a(0) = 1$ and i_n is constant for all n.

(a) Prove that $a(t) = (1 + i)^t$ for all integers $t \geq 0$.

(b) Can you conclude that $a(t) = (1 + i)^t$ for all $t \geq 0$?

1-6. Let $A(t)$ be an amount function. For every positive integer n, define $I_n = A(n) - A(n-1)$.

(a) Explain verbally what I_n represents.

(b) Prove that $A(n) - A(0) = I_1 + I_2 + \cdots + I_n$.

(c) Explain verbally the result in part (b).

(d) Is it true that $a(n) - a(0) = i_1 + i_2 + \cdots + i_n$? Explain.

1-7. (a) In how many years will 1000 accumulate to 1400 at 4% simple interest?

(b) At what rate of simple interest will 1000 accumulate to 1500 in 12 years?

(c) Repeat parts (a) and (b) assuming compound interest instead of simple interest.

1-8. At a certain rate of simple interest, 1000 will accumulate to 1060 after a certain period of time. Find the accumulated value of 500 at a rate of simple interest $\frac{2}{3}$ as great over twice as long a period of time.

1-9. Find the accumulated value of 6000 invested for ten years, if the compound interest rate is 3% per year for the first four years and 4.2% per year for the last six.

1-10. Annual compound interest rates are 4.3% in 2016, 3.7% in 2017 and 5% in 2018. Find the effective rate of compound interest per year which yields an equivalent return over the three-year period.

1-11. At a certain rate of compound interest, it is found that 1 grows to 2 in x years, 2 grows to 3 in y years, and 1 grows to 5 in z years. Prove that 6 grows to 10 in $z - x - y$ years.

1-12. If 1 grows to K in x periods at compound rate i per period and 1 grows to K in y periods at compound rate $2i$ per period, which one of the following is always true? Prove your answer.

(a) $x < 2y$

(b) $x = 2y$

(c) $x > 2y$

(d) $y = \sqrt{x}$

(e) $y > 2x$

1.4 Present Value and Discount

1-13. Henry has an investment of 1000 on January 1, 2023 at a compound annual rate of discount $d = .04$.

(a) Find the value of his investment on January 1, 2020.

(b) Find the value of i corresponding to d.

(c) Using your answer to part (b), rework part (a) using i instead of d. Do you get the same answer?

1-14. Mary has 14,000 in an account on January 1, 2020.

(a) Assuming compound interest at 4% per year, find the present value on January 1, 2014.

(b) Assuming compound discount at 4% per year, find the present value on January 1, 2014.

(c) Explain the relative magnitude of your answers to parts (a) and (b).

1-15. (a) Sketch a graph of $a(t)$ with its extension to present value in the case of simple interest.

(b) Explain, both mathematically and verbally, why $1-it$ is not the correct present value t years in the past, when simple interest is assumed.

1-16. Prove that d_n is constant in the case of compound interest.

1-17. Prove each of the following identities mathematically. For parts (a), (b) and (c), give a verbal explanation of how you can see that they are correct.

(a) $d = iv$

(b) $d = 1 - v$

(c) $i - d = id$

(d) $\frac{1}{d} - \frac{1}{i} = 1$

(e) $d\left(1 + \frac{i}{2}\right) = i\left(1 - \frac{d}{2}\right)$

(f) $i\sqrt{1-d} = d\sqrt{1+i}$

1-18. Four of the following five expressions have the same value (for $i > 0$). Which one is the exception?

(a) $\frac{d^3}{(1-d)^2}$

(b) $\frac{(i-d)^2}{1-v}$

(c) $(i-d)d$

(d) $i^3 - i^3 d$

(e) $i^2 d$

1-19. The interest on L for one year is 210. The equivalent discount on L for one year is 200. What is L?

1.5 Nominal Rate of Interest

1-20. Acme Trust offers three different savings accounts to an investor.

Account A compound interest at 4.1% per year convertible quarterly.

Account B compound interest at 4.096% per year convertible 5 times per year.

Account C compound discount at 4.064% per year convertible 10 times per year.

Which account is most advantageous to the investor? Which account is most advantageous to Acme Trust?

1-21. Phyllis takes out a loan of 3000 at a nominal rate of 5.4% per year convertible 6 times a year. How much does she owe after 22 months?

1-22. The Bank of Newfoundland offers a 4.8% mortgage convertible semiannually. Find each of the following:

(a) i

(b) $d^{(4)}$

(c) $i^{(12)}$

(d) The equivalent effective rate of interest per month.

1-23. 100 grows to 102.50 in 6 months. Find each of the following:

(a) The effective rate of interest per half-year.

(b) $i^{(2)}$

(c) i

(d) $d^{(3)}$

1-24. Jack deposits 20 into a fund today and 30 twenty years later. The fund earns interest at a nominal rate of d compounded quarterly for the first 15 years, and at a nominal rate of 4% compounded semiannually thereafter. The fund accumulates to 150 at the end of 40 years. Find d.

1-25. Daya deposits 500 into a savings account which pays interest at a nominal rate of i compounded quarterly. At the same time Pramila deposits 1000 into a different savings account which pays simple interest at an annual rate of i. Both accounts earn the same amount of interest during the first 3 months of the 12^{th} year. Find i.

1-26. Find n such that $1 + \frac{i^{(n)}}{n} = \frac{1+\frac{i^{(6)}}{6}}{1+\frac{i^{(8)}}{8}}$.

1-27. Express $d^{(7)}$ as a function of $i^{(5)}$.

1-28. Show that $v\left(1 + \frac{i^{(3)}}{3}\right) = \left(1 + \frac{i^{(30)}}{30}\right)\left(1 - \frac{d^{(5)}}{5}\right)\sqrt{1-d}$.

1-29. Prove that $i^{(4)}d^{(8)} \geq i^{(8)}d^{(4)}$.

1-30. (a) Prove that $i^{(m)} - d^{(m)} = \frac{i^{(m)}d^{(m)}}{m}$.

(b) Prove that $\frac{1}{d^{(m)}} - \frac{1}{i^{(m)}} = \frac{1}{m}$.

1.6 Force of Interest

1-31. Find the equivalent value of δ in each of the following cases.

(a) $i = .043$

(b) $d = .043$

(c) $i^{(4)} = .043$

(d) $d^{(5)} = .043$

1-32. In Section 1.3, it was shown that for $0 < t < 1$, $(1+i)^t < 1 + it$. Show that $1 + it - (1+i)^t$ is maximized at $t = \frac{1}{\delta}[\ln i - \ln \delta]$.

1-33. Assume that the force of interest is doubled.

(a) Show that the effective annual interest rate is more than doubled.

(b) Show that the effective annual discount rate is less than doubled.

1-34. Show that $\lim\limits_{i \to 0} \frac{i-\delta}{\delta^2} = .50$.

1-35. Find $a(t)$ if $\delta_t = .04(1+t)^{-1}$.

1-36. Obtain an expression for δ_t if $A(t) = ka^{t+1}b^{t^3}c^{d^t}$.

1-37. Janet makes payments on a loan at a continuous rate of $tk + 18k$ where $0 \leq t \leq 20$. The interest rate on the loan is a force of interest given by $\delta_t = \frac{1}{t+18}$. After 10 years, the accumulated value of all of her payments is 5,000.

(a) Find k.

(b) Find the accumulated value of all of her payments after 20 years.

1-38. Using mathematical induction, prove that for all positive integers n, $\frac{d^n}{dv^n}(v^{n-1}\delta) = -(1+i)(n-1)!$.

1-39. Express v as a power series expansion in terms of δ.

1-40. Express d as a power series expansion in terms of i.

1-41. Prove that $i^{(n)} < i^{(m)}$ if $n > m$.

1-42. Prove that $d < d^{(n)} < \delta < i^{(n)} < i$ for all $n > 1$.

1-43. Show that $\frac{d}{dt}(\delta_t) = \frac{\frac{d^2}{dt^2}A(t)}{A(t)} - (\delta_t)^2$.

1-44. Show that $\delta = \frac{d+i}{2} + \frac{d^2-i^2}{4} + \frac{d^3+i^3}{6} + \cdots$.

1-45. Which is larger, $i - \delta$ or $\delta - d$? Prove your answer.

2

Interest: Basic Applications

2.1 Equation of Value

In its simplest terms, every interest problem involves only four quantities: the principal originally invested, the accumulated value at the end of the period of investment, the period of investment, and the rate of interest. Any one of these four quantities can be calculated if the others are known.

In this section we will present a number of examples illustrating the determination of principal, accumulated value, and period of investment; determining the rate of interest will be explored in Sections 2.2 and 2.3. More complicated situations involving several "principals" invested at different times will arise in practice, and we will examine some of these as well.

The most important tool in dealing with such problems is the time diagram, which we encountered in chapter one, and the first step in any solution should be to draw such a diagram. After that, all entries on the diagram should be "brought" to the same point in time, in order that they can be compared. Then an equation of value is set up at that point in time, and a solution is obtained by algebraic means. The student should carefully study the examples in this section to see how these steps are carried out in practice.

We remark that before calculators came into general use, the calculations involved in some of these problems were quite difficult, and it was necessary to develop a collection of techniques to deal with them. Interest tables and log tables were in frequent use, and values which did not appear in the interest tables were handled by interpolation or other approximate methods. For example, the power series expansions given in the previous chapter could be used for calculation, since the first few terms often give a good approximation to the correct answer.

Of course we will not need to employ the older techniques. That does not mean that every question can be solved by pushing the appropriate button, however; in particular we will see cases where some approximate method (e.g., linear interpolation) is required to obtain an answer. In addition it is often necessary to first analyze the data very carefully, and organize it in such a way such that the calculator can then be called upon to assist in solving the problem. After all, your calculator is only an aid to mechanical computation. The person with the problem still has to solve it!

Example 2.1. Find the accumulated value of 500 after 173 months at a rate of interest of 4% convertible quarterly, assuming compound interest throughout.

Solution. The effective rate of interest is .01 per 3 month period, and there are a total of $57\frac{2}{3}$ periods.
Hence the answer is $500(1.01)^{173/3} = 887.50$. □

Remarks

1. It is sometimes convenient to assume compound interest over integral durations, but simple interest between integral durations. Under that assumption, the answer to this example would be $500(1.01)^{57}\left[1 + (.01)(\frac{2}{3})\right] = 887.51$. Observe that this answer is larger than the one in the example, agreeing with our earlier observation that simple interest gives a higher return when the period is less than a year.

2. In pre-calculator days the calculation of $500(1.01)^{173/3}$ would require some work. Log tables, if available, could give the answer quickly but if only interest tables were available, you might have to write the product as $500(1.01)^{50}(1.01)^{7}(1.01)^{2/3}$. The values of $(1.01)^{50}$ and $(1.01)^{7}$ could be found in the interest tables, in particular in the $n = 50$ and $n = 7$ rows of the $i = 1\%$ table. There is no $n = 57$ row of most interest tables, which is why $(1.01)^{57}$ would have to be broken up into two parts. The term $(1.01)^{2/3}$ presents a special problem. Usually only integral values of n are given in the interest tables, along with common fractional values such as $\frac{1}{2}$, $\frac{1}{4}$ and $\frac{1}{12}$, but not $\frac{2}{3}$. One could work this out by observing that $(1.01)^{2/3} = [(1.01)^{1/12}]^{8}$, but otherwise log tables or a power series expansion would be required.

Example 2.2. Alice borrows 5000 from The Friendly Finance Company at a rate of interest of 6% per year convertible semiannually. Two years later she pays the company 3000. Three years after that she pays the company 2000. How much does she owe seven years after the loan is taken out?

Solution. We will use a time diagram to aid in our solution:

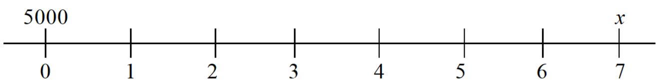

Figure 2.1

Let X be the amount still owed. In this type of problem, our goal is to obtain an equation of value which will yield the solution. To do that, all entries on the time diagram should be brought to the same point in time so an equation can be found. Any point in time can be chosen, but the most convenient one in this example is $t = 7$. The amount owed will equal the accumulated value at time 7 of the loan, minus the accumulated value at time 7 of the payments already made. Since the actual rate of interest is .03 effective per half-year, we have

$$X = 5000(1.03)^{14} - 3000(1.03)^{10} - 2000(1.03)^{4} = 1280.18.$$

□

Example 2.3. Eric deposits 8000 in an account on January 1, 2020. On January 1, 2022, he deposits an additional 6000 in the account. On January 1, 2026, he withdraws 12000 from the account. Assuming no further deposits or withdrawals are made, find the amount in Eric's account on January 1, 2029, if $i = .03$.

Solution. In this example, we see that withdrawals can be viewed as "negative deposits" in an equation of value.

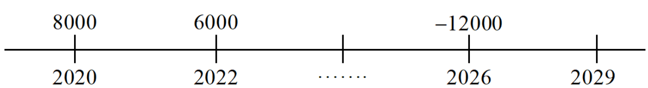

Figure 2.2

The resulting balance is

$$X = 8000(1.03)^{9} + 6000(1.03)^{7} - 12{,}000(1.03)^{3} = 4704.70.$$

□

Example 2.4. Find the net present value of Eric's deposits and withdrawals (in Example 2.3) on January 1, 2014.

Solution. In Example 2.3 we saw that the accumulated value of all deposits and withdrawals on January 1, 2029, is 4704.70. To find the net present value on January 1, 2014, we multiply by the discount factor $(\frac{1}{1.03})^{15}$ and obtain $4704.70(\frac{1}{1.03})^{15} = 3019.77$.

Note that if this question were asked directly (without first doing Example 2.3) we would obtain the answer by combining the present values of the individual deposits and withdrawals, obtaining

$$8000\left(\frac{1}{1.03}\right)^6 + 6000\left(\frac{1}{1.03}\right)^8 - 12000\left(\frac{1}{1.03}\right)^{12}$$

which gives the same answer. □

Example 2.5. John borrows 3000 from The Friendly Finance Company. Two years later he borrows another 4000. Two years after that he borrows an additional 5000. At what point in time would a single loan of 12,000 be equivalent if $i = .06$?

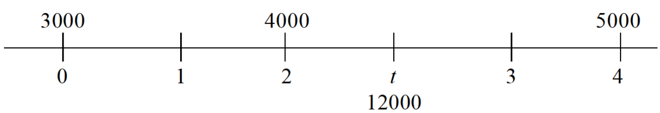

Figure 2.3

Solution. We let t be the number of years after the 3000 loan at which a single loan of 12,000 would be equivalent, and form the equation of value at time 0 as $12{,}000v^t = 3000 + 4000v^2 + 5000v^4$, where $v = \frac{1}{1.06}$. Then $v^t = \frac{3+4v^2+5v^4}{12}$. Taking logs of both sides of this equation we find $t = \frac{\ln(3+4v^2+5v^4)-\ln 12}{\ln\ v} = 2.25824$. □

There is an approximate method of solving problems like Example 2.5, called the method of equated time, but we will not need to examine it here since there are no difficulties in obtaining an exact solution.

To conclude this section, we give a very simple example where the rate of interest is the unknown.

Example 2.6. Find the rate of interest such that an amount of money will double itself over 15 years.

Solution. Let i be the required effective rate of interest. We have $(1+i)^{15} = 2$, so that $i = 2^{1/15} - 1 = .04729$. □

2.2 Unknown Rate of Interest

When the rate of interest is the unknown in an equation of value, complications often arise. To illustrate this, consider the following example.

Example 2.7. Joan deposits 2000 in her bank account on January 1, 2020, and then deposits 3000 on January 1, 2023. If there are no other deposits or withdrawals and the amount of money in the account on January 1, 2025 is 5600, find the effective rate of interest she earns.

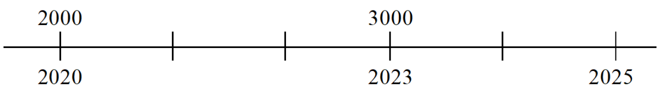

Figure 2.4

Solution. $2000(1+i)^5 + 3000(1+i)^2 = 5600$ is the equation of value on January 1, 2025. Now we have a problem. This equation is a fifth degree polynomial in i, and there is no exact formula for finding its solution. Most students will have a subroutine available on their calculators which will enable them to approximate the answer with a high degree of accuracy, obtaining $i = .035619$. $\square$

2.3 Time-Weighted Rate of Return

The rate of interest calculated in Section 2.2 is often called the dollar-weighted rate of investment return. A very different procedure is used to calculate the time-weighted rate of investment return, and that is what we will consider here. We remark before starting that in this section the compound interest assumption is no longer being made.

To calculate the time-weighted rate of return, it is necessary to know the accumulated value of an investment fund just before each deposit or withdrawal occurs. Let B_0 be the initial balance in a fund, B_n the final balance, $B_1, \ldots, B_{n-1}$ the intermediate values just preceding deposits or withdrawals, and $W_1, \ldots, W_{n-1}$ the amount of each deposit or withdrawal, where $W_i > 0$ for deposits and $W_i < 0$ for withdrawals. Let $W_0 = 0$. Then

$$i_t = \frac{B_t}{B_{t-1} + W_{t-1}} - 1 \tag{2.1}$$

represents the rate of interest earned in the time period between balances B_{t-1} and B_t. The time-weighted rate of return is then defined by

$$i = (1+i_1)(1+i_2)\cdots(1+i_n) - 1. \tag{2.2}$$

Example 2.8. On January 1, 2019, Graham's stock portfolio is worth 500,000. On April 30, 2019, the value has increased to 520,000. At that point, Graham adds 60,000 worth of stock to his portfolio. Six months later, the value has dropped to 560,000, and Graham sells 80,000 worth of stock. On December 31, 2019, the portfolio is again worth 500,000. Find the time-weighted rate of return for Graham's portfolio during 2019.

Solution. The accumulation rate from January 1 to April 30 is given by the factor $1 + i_1 = \frac{520{,}000}{500{,}000} = 1.04$. Immediately after the April 30 stock purchase, the portfolio is worth 580,000. Hence the accumulation rate from May 1 to October 31 is $1 + i_2 = \frac{560{,}000}{580{,}000} = .96552$. Finally, the accumulation rate in the last two months of the year is $1 + i_3 = \frac{500{,}000}{480{,}000} = 1.04167$.
The time-weighted rate of return for the year is found from the interval accumulation factors as $i = (1.04)(.96552)(1.04167) - 1 = .045977$. □

Note in Example 2.8 that the value of the portfolio decreased during the period from May 1 to October 31, so we see that compound interest is clearly not operating here. Nevertheless, it is still possible to calculate a dollar-weighted rate of return by considering only deposits and withdrawals, and ignoring intermediate balances. Setting up the equation of value by accumulating all quantities to December 31, 2019, we obtain

$$500{,}000(1+i) + 60{,}000(1+i)^{2/3} - 80{,}000(1+i)^{1/6} = 500{,}000.$$

This could be solved by the techniques of Section 2.3, but a popular alternative approach to this type of problem is to assume simple interest for periods less than a year. We would then obtain

$$500{,}000(1+i) + 60{,}000(1+\frac{2}{3}i) - 80{,}000(1+\frac{1}{6}i) = 500{,}000.$$

Since this equation is linear in i, the result $i = \frac{60{,}000}{1{,}580{,}000} = .03797$ is easily obtained. It should always be assumed that this alternative approach is the appropriate one to use when the interest period in question is less than or equal to one year.

EXERCISES

2.1 Equation of Value

2-1. Brenda deposits 7000 in a bank account. Three years later, she withdraws 5000. Two years after that, she withdraws an additional 3000. One year after that, she deposits an additional 4000. Assuming $i = .04$, and that no other deposits or withdrawals are made, how much is in Brenda's account ten years after the initial deposit is made?

2-2. Eileen borrows 2000 on January 1, 2022. On January 1, 2023, she borrows an additional 3000. On January 1, 2026, she repays 4000. Assuming $i = .043$, how much does she owe on January 1, 2030?

2-3. Boswell wishes to borrow a sum of money. In return, he is prepared to pay as follows: 200 after 1 year, 500 after 2 years, 500 after 3 years and 700 after 4 years. If $i = .042$, how much can he borrow?

2-4. Payments of $800, 500$ and 700 are made at the ends of years $2, 3$ and 6 respectively. Assuming $i = .044$, find the point at which a single payment of 2100 would be equivalent.

2-5. A vendor has two offers for a house:

(i) 40,000 now and 40,000 two years hence, or

(ii) 28,000 now, 24,500 in one year, and 27,500 two years hence.

He makes the remark that one offer is "just as good" as the other. Find the two possible rates of interest which would make his remark correct.

2-6. (a) The present value of 2 payments of 1000 each, to be made at the end of n years and $n + 4$ years, is 1250. If $i = .04$, find n.

(b) Repeat part (a) if the payments are made at the end of n years and $4n$ years.

2-7. In return for payments of 400 at the end of 3 years and 700 at the end of 8 years, a woman agrees to pay X at the end of 4 years and $2X$ at the end of 6 years. Find X if $i = .047$.

2-8. How long should 1000 be left to accumulate at $i = .04$ in order that it may amount to twice the accumulated value of another 1000 deposited at the same time at 3% effective?

2-9. Fund A accumulates at 4.5% effective and Fund B at 4% effective. At the end of 10 years, the total of the two funds is 52,000. At the end of 8 years, the amount in Fund B is three times that in Fund A. How much is in Fund A after 15 years?

2-10. John pays Henry 500 every March 15 from 2026 to 2030 inclusive. He also pays Henry 300 every June 15 from 2028 to 2031 inclusive. Assuming $i^{(4)} = .057$, find the value of these payments on

(a) March 15, 2035;

(b) March 15, 2029;

(c) March 15, 2025.

2.2 Unknown Rate of Interest

2-11. Alex is offered a payment of 500 at time 10. Alternatively, he is offered payments of 200 at time 0, 100 at time n and 200 at time $2n$. If $v^n = 0.82$ and the present values of the two payment options are equal, find i.

2-12. A consumer purchasing a refrigerator is offered two payment plans:
Plan A: 150 down, 200 after 1 year, 250 after 2 years
Plan B: 87 down, 425 after 1 year, 50 after 2 years
Determine the range of interest rates for which Plan A is better for the consumer.

2-13. Find the effective rate of interest if payments of 300 at the present, 200 at the end of one year, and 100 at the end of two years accumulate to 680 at the end of three years.

2-14. Bernie borrows 5000 on January 1, 2020, and another 5000 on January 1, 2023. He repays 3000 on January 1, 2022, and then finishes repaying his loans by paying 8,000 on January 1, 2025. What effective annual rate of interest is Bernie being charged?

2-15. John buys a TV for 600 from Jean. John agrees to pay for the TV by making a cash down payment of 50, then paying 100 every four months for one year (i.e. three payments of 100), and finally making a single payment 16 months after the purchase (i.e. four months after the last payment of 100).

(a) Find the amount of the final payment if John is charged interest at an effective rate of 5% per year.

(b) Find the effective annual interest rate if John's final payment is 270.

2-16. A trust company pays 3% effective on deposits at the end of each year. At the end of every four years, a 2% bonus is paid on the balance at that time. Find the effective rate of interest earned by an investor if he leaves his money on deposit for

(a) 3 years;

(b) 4 years;

(c) 5 years.

2-17. The present value of a series of payments of 1 at the end of every 2 years forever is equal to $\frac{256}{33}$. Find the effective rate of interest per year.

2.3 Time-Weighted Rate of Return

2-18. Emily's trust fund has a value of 100,000 on January 1, 2020. On April 1, 2020, 6,000 is withdrawn from the fund, and immediately after this withdrawal the fund has a value of 97,000. On January 1, 2021, the fund's value is 104,000.

(a) Find the time-weighted rate of investment return for this fund during 2020.

(b) Find the dollar-weighted annual rate of investment return for Emily's fund, assuming simple interest.

(c) Find the rate of return for Emily's fund using simple interest, and assuming a uniform distribution throughout the year of all deposits and withdrawals.

2-19. Assume in Question 18 that, in addition to the information given, there is also a 3000 deposit to the fund on July 1, 2020.

(a) Find the dollar-weighted annual rate of investment return for the fund, assuming simple interest.

(b) Find the rate of return for Emily's fund using simple interest and assuming a uniform distribution throughout the year of all deposits and withdrawals.

(c) Is it possible to calculate the time-weighted rate of return? If not, why not?

2-20. Jeffrey's investment account has a value of 10 on January 1, 2016. On July 1, 2016, Jeffrey withdraws X from the account. Immediately before this withdrawal, the value of the account is 10.5. On December 31, 2016, the value of the account is $\frac{1}{3}X$. The time-weighted rate of return for this account is 0. Assuming simple interest, find the dollar-weighted rate of return.

2-21. Daniel's investment account has a value of 100 on January 1, 2015. On July 1, 2015, his account has a value of 110, and immediately thereafter he makes a withdrawal of X. On October 1, 2015, he makes a deposit of $2X$ — immediately before this deposit his account has a value of 101. On December 31, 2015, the value of the account is 110.

Bernard's investment account also has a value of 100 on January 1, 2015. On July 1, 2015, his account has a value of 110, and immediately thereafter he makes a withdrawal of X. On December 31, 2015, Bernard's account has a value of 101.60.

During 2015, the dollar-weighted rate of return (assuming simple interest) i for Daniel's account equals the time-weighted rate of return for Bernard's account. Find i.

2-22. Let A be the balance in a fund on January 1, 2019, B the balance on June 30, 2019, and C the balance on December 31, 2019.

(a) If there are no deposits or withdrawals, show that the dollar-weighted and time-weighted rates of return for 2019 are both equal to $\frac{C-A}{A}$.

(b) If there was a single deposit of W immediately after the June 30 balance was calculated, find expressions for the dollar-weighted and time-weighted rates of return for 2019. (Assume simple interest for periods of less than a year.)

(c) If there was a single deposit of W immediately before the June 30 balance was calculated, find expressions for the dollar-weighted and time-weighted rates of return for 2019. (Assume simple interest for periods of less than a year.)

(d) Give a verbal explanation for the fact that the dollar -weighted rates of return in parts (b) and (c) are equal.

(e) Show that the time-weighted rate of return in part (b) is larger than the time-weighted rate of return in part (c).

EXTENDED SPREADSHEET EXERCISES

1. In worksheet 2-1, use the cash flows in Example 2.3, deposit 8000 on 1/1/2020; deposit 6000 on 1/1/2022; withdraw 12,000 on 1/1/2026. Complete the table in EXCEL showing the annual increments in time and the date in columns B and C, respectively. Show deposits and withdrawals at the specified time in column D and write a formula for column E that references the interest rate in CELL B1 and calculates the account balance at the beginning of each year from 1/1/2020 to 1/1/2029.

	A	B	C	D	E
1	Interest Rate	3%			
2					
3		Time	Date	D/W	Balance
4		0	1/1/20	8000	8000.00
5		1	1/1/21		8240.00
6		2	1/1/22	6000	14487.20
7		3			

Figure 2.5: Image of worksheet 2-1

a. Assuming no further deposits or withdrawals are made, find the amount in Eric's account on 1/1/2029 if $i = 0.03$.

b. Assuming no further deposits or withdrawals are made, find the amount in Eric's account on 1/1/2029 if $i = 0.06$.

c. Assume deposits of 8000 on 1/1/2020 and 1/1/2021; deposits of 6000 each January 1 from 2022 to 2025; withdrawals of 12,000 each January 1 from 2026 to 2029. Find the amount in Eric's account on 1/1/2029 if $i = 0.05$.

2. In worksheet 2-2, use the loan amounts in Example 2.5 with time 0 on 1/1/2010. Borrow 3000 on 1/1/2010; borrow 4000 on 1/1/2012; borrow 5,000 on 1/1/2014. Complete the table in EXCEL showing the annual increments in time and the date in Columns B and C, respectively. Show the amounts borrowed at the specified times in Column D. Create the annual discount factors in Column E. Create a formula in CELL B2 that references the interest rate in CELL B1 and calculates the point in time where a single loan of the total amount of 12,000 borrowed is equivalent at the specified interest rate.

	A	B	C	D	E
1	Interest Rate	6%			
2	Equivalent Time		Mthd of Equated Time		
3		Time	Date	Loan	Discount Factors
4		0	1/1/10	3000	1.00
5		1	1/1/11		0.94

Figure 2.6: Image of worksheet 2-2

a. Calculate the equivalent time assuming as in Example 2.5 $i = 0.06$.

b. Calculate the equivalent time assuming $i = 0.18$; how did the large change in interest rate affect the equivalent time for a single loan?

c. Calculate the equivalent time for a single loan using the method of equated time. This method simply takes the weighted average of the time the loan amounts are made by using the loan amounts as the weights:

$$\bar{t} = \frac{\sum t_k C_k}{C}$$

Where the loan amount at time t_k is C_k and the total loan amount is $C \sum C_k$. Place this formula in CELL E2.

3. In worksheet 2-3, use the cash flow in Example 2.8 with time 0 on 1/1/2019. The initial balance of 500,000 should be input as a deposit in column D on the date 1/1/2019. The balance before contributions is given in column E and the deposit (+) or withdrawal (-) is recorded in column D: balance b/f W/D 525,000 on 5/1/2019; deposit 50,000 on 5/1/2019; balance b/f W/D 560,000 on 11/1/2019; withdraw 100,000 on 11/1/2019; finally, balance b/f W/D 500,000 on 1/1/2020.

	A	B	C	D	E	F	G	H	I	J	K
1											
2						Dollar Weight	0.09834936				
3						Time Weighte	0			Exact Dollar- Weighted	
4											
5		Time	Date	D/W	Bal b/f Contribu	Used to calculate T-W Returr		PV of D/W			
6		0	1/1/19	500,000	500,000	500,000	1	500,000			
7		1	2/1/19			500000	1	-			
8		2	3/1/19			500000	1	-			
9		3	4/1/19			500000	1	-			

Figure 2.7: Image of worksheet 2-3

a. When cash flows and balances have been recorded, run the Exact Dollar-Weighted macro by clicking the macro button. Compare your results to the results in Example 2.8.

b. Delete the May 1 deposit and balance. What affect did this have on the time-weighted and dollar weighted return?

c. Now record the previous May 1 deposit and balance on July 1; that is balance b/f W/D 525,000 on 7/1/2019; deposit 50,000 on 7/1/2019 and run the macro. How do your results compare with part (a), explain?

d. Solve Exercise 2-18 using the spreadsheet. Compare the simple interest approximations to the dollar weighted return with exact dollar-weighted return obtained using the macro.